Decision Making with Uncertainty: Applications to Queueing Theory and Market Research

by

Matthew L. Potoff

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Doctoral Committee:

Assistant Professor Hyun-Soo Ahn, Co-Chair
Assistant Professor Damian R. Beil, Co-Chair
Assistant Professor Hila Etzion
Associate Professor Mark E. Lewis, Cornell University

It will all be worth it when it is over.
MLP, May 1993

To My Family

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Chapter 1

Introduction

This dissertation represents innovative work in the field of decision making with uncertainty. The intention of this work is to provide novel managerial insights to decision making problems that confront many firms.

In the first part of this work, a firm with two servers faces the decision-making problem of how to route arriving customers to two separate queues. Additionally, the firm has the ability to dynamically assign its two servers to whichever queue it sees fit. The servers may work together or separately, and we provide insights into how the firm should optimally route customers and assign servers to queues. The systems we analyze are subject to random customer arrivals and service times with the uncertainty in these systems being drawn from these two areas.

In the second part of this work, the firm seeks to optimize a pricing decision for an innovative product. Since the product is new, the firm has limited information on the willingness of customers to pay for the item. To learn the true demand for the product, the firm holds a series of auctions to elicit consumers' valuations. The primary decision making problem the firm faces is when to stop holding auctions, where a limited number of units can be sold, and turn to a traditional fixed-price sales campaign, where a large number of units can be sold. We solve a stopping time problem for the firm, where the data input to the firm is updated consumer valuation data received from the latest auction. In this problem, the uncertainty lies in the consumer's willingness to pay. The firm knows the variability in its willingness to pay forecast and it ultimately seeks to improve the quality of the forecast and the pricing decision made from that forecast.

Our final contribution is the proposal of a new auction mechanism which will theoretically induce auction participants to reveal their valuations for a product under the assumption that the product will be available at a date in the near future for an unknown fixed price. We refer to this mechanism as a second-price auction with rebate. Additionally, we find symmetric equilibrium bidding strategies for three other mechanisms, which can be used to reverse engineer auction bid data into willingness to pay or demand curve information.

Chapter 2

Server Allocation and Customer Routing Policies for Two Parallel Queues when Service Rates are not Additive

2.1 Introduction

Consider a system of two parallel queues where customers arrive to each queue at random times. There are two servers which may be allocated to either of the queues. We assume Poisson arrivals and exponential service times. The firm has the choice of instantaneously moving the servers at no cost and/or routing an arriving customer to the other queue at some cost. The firm accepts all arrivals (i.e., rejecting customers from both queues is not allowed). We seek to characterize optimal server allocation and customer routing policies for this system.

One characteristic that distinguishes this model from previous work is that when the two servers work together at the same queue, we allow for service rates which are either superadditive or subadditive. In the case of superadditive service rates, we show that the firm allocates both servers to the higher cost queue and only services the lower cost queue when the higher cost queue is empty. We also show that the firm will route customers from the higher cost queue to the lower cost queue according to a monotonic switching curve policy. Additionally, we show that the firm will never route a customer from the lower cost queue to the higher cost queue.

In the case of subadditive service rates, we present examples that show the complexity of the optimal policy. We then prove the structure of the optimal routing policy for the case of symmetric arrival rates, service rates, and holding costs. We show that the optimal routing policy follows a switching curve, where if we route from queue 1 to queue 2 in one state, then we will also route from queue 1 to queue 2 if: (a) there is one more in line at queue 1 or (b) there is one more in line at queue 1 and one fewer in line at queue 2.

Routing Literature. Early analytical work in the area of routing to parallel queues includes Haight (1958), where the asymptotic state probabilities are found in the case of arrivals joining the shortest queue. Kingman (1961) finds the waiting time distribution for the system of symmetric queues where arrivals are routed to the shortest queue. Knessl et al. (1986) find approximations to the number of customers at the two queues when the service rates are heterogeneous. A more in-depth analysis of the properties of these systems can be found in Adan et al. (1991). Adan et al. (2001) provides a survey of analytical solution methods for queueing systems of this type.

Optimal customer routing policies have been studied extensively. Winston (1977c) and Weber (1978) seek to to maximize the number of customers served within a finite period of time. Winston (1977c) proves that routing customers to the shortest line is optimal when arrivals follow a Poisson process and servers are identical and exponential. For the case for general arrivals and identical servers, where the service time distributions have non-decreasing hazard rates, Weber (1978) extends these results to route customers to the queue with the shortest expected wait prior to entering service. Others to show that routing to the shortest queue is optimal include Hordijk and Koole (1990), Menich and Serfozo (1991), Sparaggis et al. (1996), and Koole et al. (1999). Whitt (1986) provides counterexamples to show that joining the shortest queue is not always optimal when the service time distributions are not increasing failure rate.

In Hordijk and Koole (1990), a generalized shortest queue routing policy is shown to be optimal. In their model, a general arrival process occurs, which may include batch arrivals. Upon arrival, the job must be routed to one of m parallel queues. There is a single exponential server at each queue. The servers are homogenous in that they have identical mean service times. They provide two models with the goal of maximizing the number of customers served in a finite time. In the first model, each server has a finite buffer size, but the customers arrive one at a time. They show that the value function is increasing and componentwise symmetric. Additionally, they show that the value function will increase more by increasing the number in line at a shorter queue than by increasing the number in line at a longer queue. Using these properties, they are able to show that it is always better to route customers to the shortest queue. In the second model, they assume infinite buffers, but customers arrive in batches which must be assigned to the same queue. The size of the batch is not known until after it is assigned to a queue. Similar properties are used to show that routing to the shortest to queue is optimal.

Ephremides et al. (1980) also show that routing to the shortest queue is optimal if the controller has full information on the state of the system. If the controller has limited information, Ephremides et al. show that a round robin policy is optimal. They also show that both the optimal policy and round robin policies are superior to Bernoulli splitting (i.e., randomly assigning arrivals to a queue). While we know Bernoulli splitting to be suboptimal, Koole (1996) proves that if the servers are homogeneous, then splitting equally between them is the best splitting policy. For early work on deterministic and non-deterministic splitting, see Chow and Kohler (1979).

Xu and Zhao (1996) study the problem of two parallel queues with a single Poisson arrival stream. Upon arrival, the customer is routed (at no cost) to one of the two queues. Customers may be transferred amongst queues at a positive cost. Holding costs and service rates may be heterogeneous in their model. Xu and Zhao prove a switching curve structure for when it is optimal to route and jockey customers.

They also prove that it is not optimal for jockeying of a customer from a low cost queue to a high cost queue unless the high cost queue is empty. Other papers which find switching curve policies for routing in parallel queues with no routing costs include Larsen and Agrawala (1983), Lin and Kumar (1984), Viniotis and Ephremides (1988), Xu et al. (1992), and Hordijk and Koole (1992). In Koyanagi and Kawai (1995), the optimal routing policy for the routing problem is also found to follow a switching curve structure when there is a positive routing cost. When there are multiple heterogeneous servers, Derman et al. (1980) find that it is optimal to assign arriving customers to the fastest open server available. They assume that customers which arrive when all servers are busy are lost. This result is also reinforced in the case of heavy traffic in Armony (2005). In the case of heterogeneous parallel servers and multiple customer types, Winston (1977b) and Winston (1977a) show that customers with longer expected service times should be routed to faster servers.

As a generalization of Hordijk and Koole (1990), optimal routing policies to m parallel queues with a single exponential server at each queue is studied in Hordijk and Koole (1992). In the more recent paper, Hordijk and Koole allow for the exponential servers to have different mean service times. They prove that the customers should be routed to a faster server when that server has a shorter queue. This only partially characterizes the optimal policy. That is, they are able to say which queue a customer should be routed to only if the fastest queue already has fewest customers upon the new arrival. If the fastest queue has more customers waiting, no structure is given. The method of proof, using the dynamic programming approach with value iteration, is the same technique that we use in this study.

More recently, analysis of routing in systems with parallel servers has turned to performance measures in heavy traffic. See Laws (1992), Harrison (1998), Harrison and López (1999), Bell and Williams (2001), Armony (2005), Down and Wu (2006), He and Down (2008), and Wu and Down (2008) for relevant work in this area. Our interest in heavy traffic is limited to the numerical study where we show that under heavy traffic the relative value of routing is much more important than server flexibility when service rates for pooled servers are subadditive.

Hariharan et al. (1990) considers the optimal routing and admission control policies for two parallel queues with an infinite number of identical exponential servers working at each queue. Similar to our model, they allow for asymmetric customer holding costs. However, they assume that the service rate at each queue is a linear combination of the service rates of the servers working at the queue. The routing and admission policies are shown to follow a switching curve. Xu (1994) also studies the admission and scheduling of arrivals at nonidentical servers with the intention of approximating the optimal thresholds. For surveys of the literature on the control of admission, routing, and server allocation in parallel queues see Crabill et al. (1977), Stidham and Weber (1993), and Kelly and Laws (1993).

While we do not consider the admission control problem, the routing monotonicity results in all of the above papers are analogous to the routing threshold policy that we present for the case of superadditive service rates (Theorem 1). We provide an example for subadditive service rates which does not follow this monotonic routing threshold policy. In the symmetric subadditive case, we are able to prove the structure of the routing policy, which is not necessarily monotonic.

Server Allocation Literature. The literature on choosing optimal service rates is also vast. Bell (1980) proves the optimal structure for the number of servers to utilize in an $M/M/c$ queueing system based on the state of the system. Sobel (1982) provides conditions under which full service policies are optimal. Weber and Stidham (1987) proves the optimality of service rate policies which are monotonic, while a unified approach towards proving the structure of the optimal service rate policy is presented in Stidham and Weber (1989). In these papers, there is a cost for servers which increases as service rates are increased.

Smith and Whitt (1981) and Calabrese (1992) prove that pooling of queues results in more efficient queueing systems. Their results are consistent with our findings for additive and superadditive service rates when servers are pooled. Since we always pool our servers in the superadditive case, our model is similar to a polling system (see Browne and Yechiali 1989 or Liu et al. 1992) with the exception that we allow for routing of customers. Rajan and Agrawal (1996) provides a way to improve the performance of a polling system where customers are routed to different queues which will be attended to by a single server.

In our paper, the use of flexible servers overlaps with the literature on the use of flexible servers in multiclass queues. Since we allow for server pooling and costless server switching, our model is similar to the service of multiclass queues, with the caveat that we also allow for customer routing. In Harrison (1975b) and Harrison (1975a), optimal service polices and expressions for the profit are found for priority queues with Poisson arrivals with arbitrary service distributions.

In Buyukkoc et al. (1985), the optimality of the $c\mu$ rule is established for discrete-time queueing systems where preemption of service is allowed and the service time distribution is geometric. The rule says that when there are N queues with holding costs $c_i, i = 1, \ldots, N$, which are served by a single server with geometric service times with mean $1/\mu_i$, it is optimal to serve the queues with the highest value of $c_i\mu_i$ first. The proof of the rule follows from a simple interchange argument.

The allocation of flexible servers has received more attention in recent literature. Green (1985) approximates the waiting time distribution for a system of parallel, heterogeneous servers with customers of two classes. One set of servers can only serve one customer type while the other set of servers is flexible and can serve both customer types. Ahn et al. (2004) analyzes the problem of how to allocate a flexible server between two parallel queues in order to clear the system at the lowest possible cost, where the cost is a linear holding cost per unit time for each job in the system.

There are two types of jobs and two types of servers. The first server can only work on jobs in his queue. The second server can work on both types of jobs, but at different rates. They present conditions under which three different policies are optimal when there are no arrivals and then show via a numerical study that these policies are near optimal under light or moderate traffic. Andradóttir et al. (2003) prove that generalized round-robin policies will perform well in systems with flexible servers and non-zero switching times for servers to change which queue they serve.

Much of the research on flexible servers is in the area of serial, or tandem, queues. Ahn et al. (1999) consider a tandem queueing system in which the servers may be moved instantaneously between the queues. The goal is to minimize the cost to clear the system, assuming no new arrivals. In their model, the service rates are additive, but depend on which queue is being serviced. They show that, dependent upon the holding costs and service rates, it is optimal to completely clear one of the queues before beginning service on the other. This result is consistent with our result for additive service rates in §3. Andradottir et al. (2001) treat a similar model, but produce a different result. They model a two server, two station tandem queueing network with Poisson arrivals and exponential service times, but have the objective of maximizing the long-run average throughput, as opposed to minimizing holding costs. They show that it is optimal to allocate one server to each queue unless the first queue is blocked or the second queue is starved. This result is similar to the server allocation policy presented in the symmetric case of subadditive service rates presented in §4. Iravani et al. (1997) and Duenyas et al. (1998) consider the use of a single flexible server in tandem queueing system and the optimal policies therein. Farrar (1993) and Wu et al. (2006) prove optimal server assignment policies for tandem queueing systems with dedicated resources at each queue and a floating server which can work at either queue.

Analysis of flexible servers in parallel systems has received much less attention in the literature. Duenyas and Van Oyen (1995) treat the problem of parallel queues which are serviced by a single server. Each queue is has a different holding cost and the server may dynamically be assigned to any queue upon the completion of a job. Similar to our model, they also assume that the server may instantaneously move between queues. However, unlike our model, they do assign a cost for switching the server between queues but do not allow for routing of customers from one queue to another. They partially describe the optimal server allocation policy. Under the assumption of linear holding costs, c_i, they prove that it is optimal to serve the queue with the highest value of $c_i\mu_i$ until that queue is empty, although they do not suggest what the server should after that queue has been depleted. Our structural results in §3 are similar to Duenyas and Van Oyen's in that we prove that both of our servers work together at the higher cost queue until it is empty. Hofri and Ross (1987) consider parallel queues with a single server, but they also assume a setup time when the server switches between queues. They reason that the optimal server allocation policy must be of a threshold type.

Our model is most similar to a variation contained in Hajek (1984). Hajek's model has two Poisson arrival streams to two queues. A third Poisson arrival stream may be routed dynamically to either queue. There is no cost to routing an arrival, but the system pays a linear holding cost per unit time for each customer in queue. The holding cost coefficients in the model may be heterogeneous. There are two heterogeneous, exponential servers, one at each queue. A third exponential server may be allocated dynamically to either queue. The optimal routing and server allocation policies are shown to follow a threshold structure. One of the main differences between Hajek's model and the one presented here is that we allow for non-linearity in the service rates when two servers are combined. Additionally, we allow for the servers to be switched between the queues, as opposed to allowing a floating server.

Our Contribution. While most previous work has addressed either customer rout-

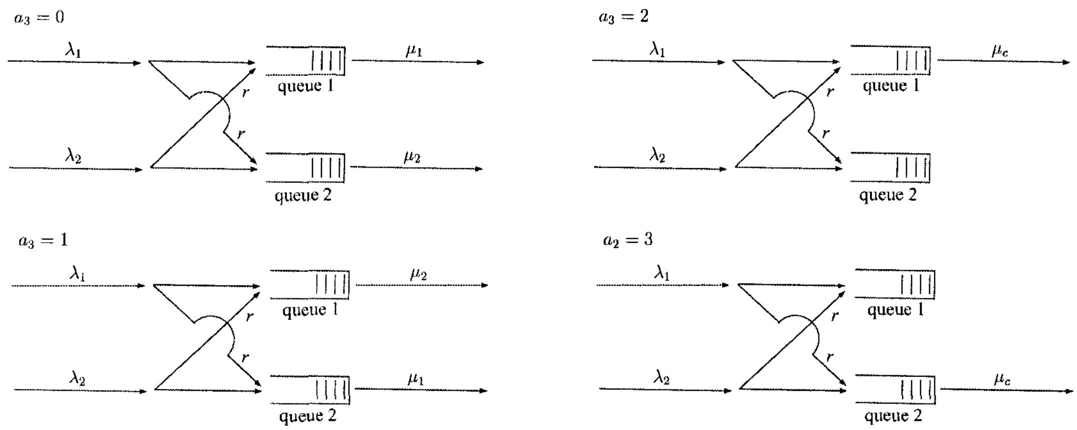


Figure 2.1: $a_3 = 0$: Allocate server 1 to queue 1, server 2 to queue 2. $a_3 = 1$: Allocate server 2 to queue 1, server 1 to queue 2. $a_3 = 2$: Allocate both servers to queue 1. $a_3 = 3$: Allocate both servers to queue 2.

ing or server allocation policies, our study addresses both. To our knowledge, this is the first paper to address both issues simultaneously in the case where worker pooling rates are not additive. Furthermore, we present a numerical study which provides insights into when each option (customer routing or flexible server allocation) is more beneficial.

The remainder of the paper is organized as follows. Section 2 describes the general model. Section 3 provides results for the case when the servers work together at a rate equal to or greater than additive. Section 4 provides structural properties of the optimal policies for the symmetric case when the servers work together at a rate less than additive. A numerical study is presented in Section 5. Lengthy proofs are contained in the Appendix.

2.2 General Model

A firm operates two service facilities (see Figure 2.1) where customers arrive at queue $i, i \in \{1, 2\}$, according to a homogeneous Poisson process with rate λ_i. Upon arrival to the queue, the firm can accept the customer at the originating queue or route the customer to the other queue for a one-time fixed payment of r. Customers can only be routed to an alternate queue upon first arrival. Routing from one queue to the other is instantaneous. The state of the system at time t is $\mathbf{s}(t) = (n_1(t), n_2(t)) \in S$, where $n_1(t)$ and $n_2(t)$ are the number of customers in queues 1 and 2, respectively. We limit $n_1(t)$ and $n_2(t)$ to the set of nonnegative integers.

Denote by $\mathbf{a} = (a_1, a_2, a_3) \in A$ the action that a firm takes at any point in time. Let a_i represent the action taken when a new customer arrives at queue $i, i = 1, 2$. We set $a_i = 0$ when the customer remains at queue i, and set $a_i = 1$ when the customer will be routed to the other queue. Let $a_3 \in \{0, 1, 2, 3\}$ represent the server allocation policy where $a_3 = 0$: allocate server 1 to queue 1 and server 2 to queue 2; $a_3 = 1$: allocate server 1 to queue 2 and server 2 to queue 1; $a_3 = 2$: allocate both servers to queue 1; $a_3 = 3$: allocate both servers to queue 2.

The firm has two servers available to work at the two queues. It has the option to allocate any server to any queue. We assume that moving a server is costless and instantaneous. When working at separate queues, the servers complete jobs according to an exponential distribution with rates μ_1 and μ_2 for servers 1 and 2, respectively, independent of which queue they are serving. That is, server 1, when working alone, works at rate μ_1 at either queue 1 or queue 2. When both servers work at the same queue, the combined service rate is μ_c. Let $\mu(\mathbf{s}, a_3)$ be the rate that the system is processing customers. That is, $\mu(\mathbf{s}, a_3) = \mu_1 + \mu_2$ if the servers are separated and $\mu(\mathbf{s}, a_3) = \mu_c$ if the servers are pooled together.

Let $Y = \{Y(t) : t \leq \infty\}$ be a controlled Markov process such that $Y(t) = Y_n$ for $T_n \leq t < T_{n+1}$ where T_n is the time of the nth jump of Y. The continuous-time controlled Markov process is defined by $Y = (S, A, c, \lambda, p, \alpha)$, where S is the state space, A is the action space, c is the cost structure, λ is the space representing the exponential transition rates, p is the space representing the transition probabilities, and $\alpha \geq 0$ is the discount rate.

The firm pays a holding cost of $h(n_1, n_2)$ per unit time when there are n_1 customers waiting in queue 1 and n_2 customers waiting in queue 2. For a given state, $\mathbf{s}(t)$, the firm chooses the actions of where to allocate servers and whether or not to route new arrivals. Let π be a deterministic Markov stationary policy and let Π be the set of all such policies. The cost during one transition takes the form of the following reward function:

$$c(\mathbf{s}, \mathbf{a}) \quad = \quad E^{\pi}_{T_1}\left[\int_0^{T_1} e^{-\alpha t} h(\mathbf{s})dt + e^{-\alpha T_1}(\rho_1(\mathbf{s}, a_1) + \rho_2(\mathbf{s}, a_2))\right].$$

The expectation is over T_1, the sojourn time in state $\mathbf{s}$ given policy π is followed. The first term inside the expectation is the discounted value of the holding cost as it is continuously accrued until the next state transition. The second term reflects the lump sum cost which is spent upon an arrival to either queue. The cost functions, $\rho_i(\mathbf{s}, a_i)$, are equal to the probability that the next state transition is an arrival at queue i times r_i if the action is to route the customer to the other queue ($a_i = 1$), and zero otherwise ($a_i = 0$).

$$\rho_i(\mathbf{s}, a_i) \quad = \quad \frac{\lambda_i}{\lambda_1 + \lambda_2 + \mu(\mathbf{s}, a_3)} \cdot r_i \cdot 1_{\{a_i = 1\}}, \quad i = 1, 2$$

Note that the transition rate out of state $\mathbf{s}$ when action, a_3, is taken is $\lambda_1 + \lambda_2 + \mu(\mathbf{s}, a_3)$. Using the optimal policy over Π, define $v(\mathbf{s})$ as the expected total discounted cost-to-go function for initial state $\mathbf{s}$ at time 0:

$$v(\mathbf{s}) = \min_{\pi \in \Pi} E^{\pi}\left[\sum_{n=0}^{\infty} e^{-\alpha T_n} c(Y_n, \mathbf{a}_n) | Y_0 = \mathbf{s}\right] \qquad (2.2.1)$$

Using uniformization (Lippman 1975, Serfozo 1979), the continuous-time controlled Markov process, $Y = (S, A, c, \lambda, p, \alpha)$, can be transformed into an equivalent discrete-time Markov decision process, $X = (S, A, c', p', \Lambda/(\alpha + \Lambda))$. First, we redefine the transition rates in such a way that the infinitesimal generator of Y is the same as for a controlled Markov process, $Y' = (S, A, c', \lambda', p', \alpha)$, with a constant (uniform) sojourn parameter, Λ, where $\Lambda = \lambda_1 + \lambda_2 + \max[\mu_c, \mu_1 + \mu_2]$. Now, define the reward function for Y' as

$$c'(\mathbf{s}, \mathbf{a}) = c(\mathbf{s}, \mathbf{a}) \frac{\alpha + \lambda_1 + \lambda_2 + \mu(\mathbf{s}, a_3)}{\alpha + \Lambda} \tag{2.2.2}$$

and the transition probabilities as

$$p'(i, \mathbf{a}, j) = \begin{cases} 1 - (\lambda_1 + \lambda_2 + \mu(i, a_3))(1 - p(i, \mathbf{a}, i))/\Lambda & \text{if } i = j \\ (\lambda_1 + \lambda_2 + \mu(i, a_3))p(i, a, j)/\Lambda & \text{if } i \neq j. \end{cases} \tag{2.2.3}$$

From Serfozo (1979), we know that the infinitesimal generators of Y and Y' are equal which implies that the two are equivalent decision processes. The process, X, can be derived by taking the expectation of Y'. Thus, X, Y', and Y are equivalent decision processes. We present the discrete-time formulation of X below.

$$v(n_1, n_2) = \frac{1}{\alpha + \Lambda} \left(\begin{array}{l} h(n_1, n_2) + \lambda_1 \min[v(n_1 + 1, n_2), v(n_1, n_2 + 1) + r] \\ +\lambda_2 \min[v(n_1 + 1, n_2) + r, v(n_1, n_2 + 1)] \\ + \min \left[\begin{array}{l} \mu_1 v((n_1 - 1)^+, n_2) + \mu_2 v(n_1, (n_2 - 1)^+) \\ +(\Lambda - \lambda_1 - \lambda_2 - \mu_1 - \mu_2) v(n_1, n_2), \\ \mu_2 v((n_1 - 1)^+, n_2) + \mu_1 v(n_1, (n_2 - 1)^+) \\ +(\Lambda - \lambda_1 - \lambda_2 - \mu_1 - \mu_2) v(n_1, n_2), \\ \mu_c v((n_1 - 1)^+, n_2) + (\Lambda - \lambda_1 - \lambda_2 - \mu_c) v(n_1, n_2), \\ \mu_c v(n_1, (n_2 - 1)^+) + (\Lambda - \lambda_1 - \lambda_2 - \mu_c) v(n_1, n_2) \end{array} \right] \end{array} \right)$$

The last term is the minimum of four expressions. The first is to have server 1 at queue 1 and server 2 at queue 2. The second is to have server 2 at queue 1 and server 1 at queue 2. The third is have both servers at queue 1. The fourth is to have both servers at queue 2. Without loss of generality we can set $\alpha + \Lambda = 1$ by scaling our time and constants appropriately. We can rewrite the above equation as

$$
\begin{aligned}
v(n_1, n_2) &= h(n_1, n_2) + \lambda_1 \min[v(n_1+1, n_2), v(n_1, n_2+1) + r] \\
&+ \lambda_2 \min[v(n_1+1, n_2) + r, v(n_1, n_2+1)] \qquad (2.2.4) \\
&+ \min \left[\begin{array}{l}
\mu_1 v((n_1-1)^+, n_2) + \mu_2 v(n_1, (n_2-1)^+) \\
+(\Lambda - \lambda_1 - \lambda_2 - \mu_1 - \mu_2) v(n_1, n_2), \\
\mu_2 v((n_1-1)^+, n_2) + \mu_1 v(n_1, (n_2-1)^+) \\
+(\Lambda - \lambda_1 - \lambda_2 - \mu_1 - \mu_2) v(n_1, n_2), \\
\mu_c v((n_1-1)^+, n_2) + (\Lambda - \lambda_1 - \lambda_2 - \mu_c) v(n_1, n_2), \\
\mu_c v(n_1, (n_2-1)^+) + (\Lambda - \lambda_1 - \lambda_2 - \mu_c) v(n_1, n_2)
\end{array} \right].
\end{aligned}
$$

We next show the intuitive result that the value function is increasing in both state variables.

Lemma 1. *If $h(n_1, n_2)$ is increasing in n_1 and n_2, then the value function, $v(n_1, n_2)$, as given in equation 2.2.4 is increasing in n_1 and n_2.*

We state the following lemma, which follows directly from the Lemma 1.

Lemma 2. *For $\mu_c > \max[\mu_1, \mu_2]$, if $n_1 = 0$ ($n_2 = 0$), then it is optimal to pool the servers at queue 2 (queue 1).*

Define the following operators on functions, $f(n_1, n_2)$, for $(n_1, n_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of nonnegative integers.

$$
\begin{aligned}
D_1 f(n_1, n_2) &= f(n_1+1, n_2) - f(n_1, n_2) \\
D_2 f(n_1, n_2) &= f(n_1, n_2+1) - f(n_1, n_2) \\
D_1^{(2)} f(n_1, n_2) &= D_1 D_1 f(n_1, n_2) \\
D_2^{(2)} f(n_1, n_2) &= D_2 D_2 f(n_1, n_2) \\
D_{21} f(n_1, n_2) &= D_{12} f(n_1, n_2) = D_1 D_2 f(n_1, n_2)
\end{aligned}
$$

$$\begin{aligned}
\Delta f(n_1,n_2) &= f(n_1+1,n_2)-f(n_1,n_2+1)\\
\Delta^{(2)} f(n_1,n_2) &= \Delta\Delta f(n_1,n_2)=\Delta f(n_1+1,n_2)-\Delta f(n_1,n_2+1)
\end{aligned}$$

These operators will be used throughout the paper.

2.3 Heterogenous Servers with Superadditive Pooling

Assume that when working at the same queue, the rate is μ_c, $\mu_c \geq \mu_1+\mu_2$. We also assume linear holding costs such that $h(n_1,n_2)=h_1n_1+h_2n_2$. Without loss of generality assume that $h_1 \geq h_2$. We present the following lemma to simplify the formulation.

Lemma 3. *If $\mu_c \geq \mu_1+\mu_2$, it is never optimal to allocate one server to each queue.*

Proof. First, assume that $v((n_1-1)^+,n_2) \leq v(n_1,(n_2-1)^+)$. That is, in state (n_1,n_2) we prefer to allocate both servers to queue 1 than to allocate both servers to queue 2. Since $v(n_1,n_2)$ is increasing in n_1 and n_2, we have that

$$\begin{aligned}
\mu_c v((n_1-1)^+,n_2) &= \mu_1 v((n_1-1)^+,n_2)+\mu_2 v((n_1-1)^+,n_2)\\
&\quad +(\mu_c-\mu_1-\mu_2)v((n_1-1)^+,n_2)\\
&\leq \mu_1 v((n_1-1)^+,n_2)+\mu_2 v(n_1,(n_2-1)^+)\\
&\quad +(\mu_c-\mu_1-\mu_2)v(n_1,n_2)
\end{aligned}$$

and

$$\begin{aligned}
\mu_c v((n_1-1)^+,n_2) &= \mu_1 v((n_1-1)^+,n_2)+\mu_2 v((n_1-1)^+,n_2)\\
&\quad +(\mu_c-\mu_1-\mu_2)v((n_1-1)^+,n_2)\\
&\leq \mu_2 v((n_1-1)^+,n_2)+\mu_1 v(n_1,(n_2-1)^+)\\
&\quad +(\mu_c-\mu_1-\mu_2)v(n_1,n_2).
\end{aligned}$$

Thus, the cost of allocating both servers to queue 1 is always less than or equal to the cost of allocating one server to each queue.

Next, assume that $v((n_1 - 1)^+, n_2) \geq v(n_1, (n_2 - 1)^+)$. That is, in state (n_1, n_2) we prefer to allocate both servers to queue 2 than to allocate both servers to queue 1. A similar argument gives us that the cost of allocating both servers to queue 2 is always less than or equal to the cost of allocating one server to each queue.

Hence, the cost of allocating both servers to a single (preferred) queue is always less than or equal to the cost of allocating one server to each queue. □

The previous lemma lets us simplify equation 2.2.4:

$$\begin{aligned} v(n_1, n_2) = h_1 n_1 + h_2 n_2 &+ \lambda_1 \min[v(n_1 + 1, n_2), v(n_1, n_2 + 1) + r] \\ &+ \lambda_2 \min[v(n_1 + 1, n_2) + r, v(n_1, n_2 + 1)] \\ &+ \mu_c \min[v((n_1 - 1)^+, n_2), v(n_1, (n_2 - 1)^+)]. \end{aligned} \tag{2.3.5}$$

We now show the structure of the optimal server allocation and customer routing policy.

Theorem 1. *If $h_1 \geq h_2$ and $\mu_c \geq \mu_1 + \mu_2$ then the optimal server allocation and customer routing policy, π^*_{sup}, has the following structure:*

- **A.** *Always pool both servers at queue 1 when $n_1 \geq 1$. If $n_1 = 0$ then pool both servers at queue 2 until $n_1 \geq 1$.*
- **B.** *Never route a customer from queue 2 to queue 1.*
- **C.** *The optimal routing policy takes the form of an increasing switching curve such that if it is optimal to route an arriving customer from queue 1 to queue 2 in state (n_1, n_2), then it is also optimal to route an arriving customer from queue 1 to queue 2 in states $(n_1 + 1, n_2)$ and $(n_1, n_2 - 1)$.*

The following lemma describes the properties of the value function, which will lead to the structure described in Theorem 1. Proof of the lemma is found in the appendix.

Lemma 4. *The value function under π^*_{sup} has the following properties for $n_1, n_2 \geq 0$:*

M1. $D_1 v(n_1, n_2), D_2 v(n_1, n_2) \geq 0$ *(increasing).*

M2. $\Delta v(n_1, n_2) \geq 0$.

M3. *(a)* $D_1 \Delta v(n_1, n_2) \geq 0$ *and (b)* $D_2 \Delta v(n_1, n_2) \leq 0$ *(diagonal dominance).*

Proof of Theorem 1

A. If $n_1, n_2 \geq 1$, the firm will prefer to allocate both servers to queue 1 instead of allocating both servers to queue 2 if $v(n_1 - 1, n_2) \leq v(n_1, n_2 - 1)$. This follows from property **M2**, $\Delta v(n_1, n_2) \geq 0$.

If $n_1 \geq 1, n_2 = 0$, then the firm allocates both servers to queue 1 since $v(n_1 - 1, 0) \leq v(n_1, 0)$ (property **M1**).

If $n_1 = 0, n_2 \geq 1$, then the firm allocates both servers to queue 2 since $v(0, n_2) \geq v(0, n_2 - 1)$ (property **M1**).

B. The firm will route a customer from queue 2 to queue 1 if and only if $v(n_1 + 1, n_2) + r \leq v(n_1, n_2 + 1)$. This is equivalent to $\Delta v(n_1, n_2) \leq -r$, which cannot occur since $\Delta v(n_1, n_2) \geq 0$ (property **M2**). Thus, the firm will never route a customer from queue 2 to queue 1.

C. The firm routes a customer from queue 1 to queue 2 in state (n_1, n_2) if and only if $v(n_1, n_2+1)+r \leq v(n_1+1, n_2)$, or $\Delta v(n_1, n_2) \geq r$. The firm routes a customer from queue 1 to queue 2 in state (n_1+1, n_2) if and only if $v(n_1+1, n_2+1)+r \leq v(n_1 + 2, n_2)$, or $\Delta v(n_1 + 1, n_2) \geq r$. Thus, property **M3**(a), $D_1 \Delta v(n_1, n_2) \geq 0$, is a sufficient condition to ensure that if the firm routes a customer from queue

1 to queue 2 in state (n_1, n_2), it will also route a customer from queue 1 to queue 2 in state $(n_1 + 1, n_2)$.

Similarly, for $n_2 \geq 1$, the firm routes a customer from queue 1 to queue 2 in state $(n_1, n_2 - 1)$ if and only if $v(n_1, n_2) + r \leq v(n_1 + 1, n_2 - 1)$, or $\Delta v(n_1, n_2 - 1) \geq r$. Thus, property **M3**(b), $D_2 \Delta v(n_1, n_2) \leq 0$ for $n_1, n_2 \geq 0$, is a sufficient condition to ensure that if the firm routes a customer from queue 1 to queue 2 in state (n_1, n_2), it will also route a customer from queue 1 to queue 2 in state $(n_1, n_2 - 1)$. $\square$

If the holding costs are symmetric, we have that $\Delta v(n_1, n_2) = 0$, which leads to this corollary.

Corollary 1. *If $h_1 = h_2$ then it is never optimal to route an arriving customer to a different queue.*

After it was shown that the servers are always pooled together when their combined service rate is superadditive, the fact that they work at the higher cost queue is not surprising, as it is consistent with the well-known $c\mu$ rule (Buyukkoc et al. 1985).

2.4 Subadditive Pooling

2.4.1 General Case

In many situations, pooling servers may result in some inefficiencies. For example, if workers become distracted by each other or if there is a shared resource which is overloaded with two workers, the overall productivity may fall. In this section we consider the case of subadditive pooling, i.e., when $\mu_1 + \mu_2 < \mu_c$.

As in the previous section, we assume linear holding costs, $h(n_1, n_2) = h_1 n_1 + h_2 n_2$. We now provide an illustrative example to show how complex the pooling and routing policies can become in the case of subadditive server pooling rates. We choose parameters: $\lambda_1 = 4.0, \lambda_2 = 5.5$ (arrival rates); $\mu_1 = 8.0, \mu_2 = 7.0, \mu_c = 14.025$ (service rates, the combined service rate is 93.5% of the sum); $r = 3$ (customer routing cost); $h_1 = 10$, $h_2 = 8$ (holding costs); $\alpha = 0.025$ (discount rate). The optimal server allocation and customer routing are shown in figure 2.2 and 2.3.

Let's observe the optimal dynamic server allocation policy. Only when n_2 is large relative the value of n_1 do we allocate the faster server to queue 2. Also, the only time we allocate both servers to queue 2 is when queue 1 is empty. However, when n_1 is large, relative to n_2, we might allocate both servers to queue 1. The choice to allocate both servers to queue 1 is only temporary. If queue 1 grows even longer, we choose to put the faster server at queue 1 and slower server at queue 2. This phenomena points to the fact that while the per unit holding cost at queue 1 is higher, the productivity loss by pooling the servers may outweigh the benefit of shortening the higher cost queue at the combined rate. Thus, it is often optimal to split the workers up when the service rates are subadditive.

This example also shows that if the workers are separated, it is not obvious to which queue each worker should be allocated. If we can dynamically move the servers, intuition suggests that the faster worker should go to the queue with the higher holding cost coefficient. However, we must also pay attention to the arrival rates. Figure 2.4 shows the server allocation policy for a similar example, except that the arrival rate at queue 2 is less than in the previous example. Based on these examples, it appears that the size of the region where the faster server is allocated to the lower cost queue increases as the arrival rate to the lower cost queue increases. It also appears that the size of the region where both servers are allocated to the higher cost queue increases as the arrival rate to the lower cost queue decreases. We conjecture that the server allocation policies follow a switching curve, but as we can see in figure 2.2, the switching curves are not necessarily monotonic.

In contrast to the routing policy in the superadditive case, when service rates are

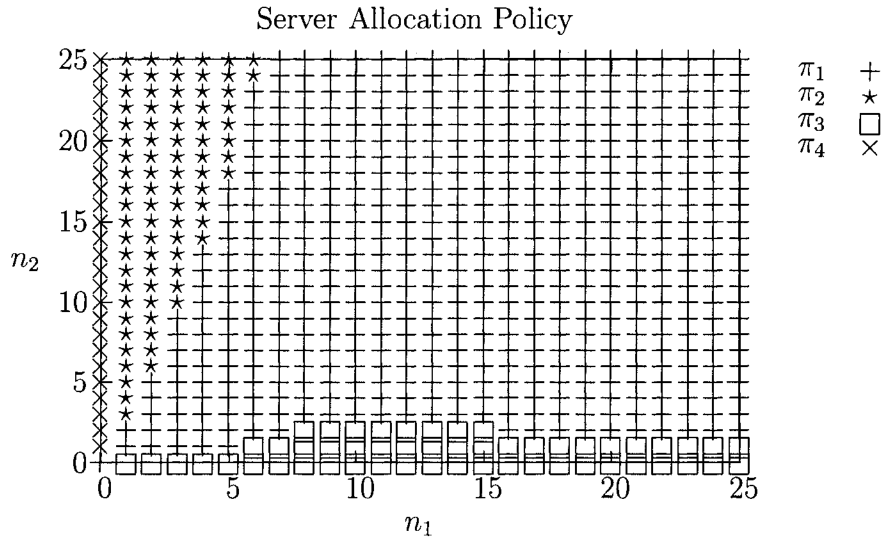


Figure 2.2: Server allocation policy for subadditive pooling rates (i). $h_1 = 10$, $h_2 = 8$, $\lambda_1 = 4$, $\lambda_2 = 5.5$, $\mu_1 = 8$, $\mu_2 = 7$, $\mu_c = 14.025$, $r = 3$, $\alpha = 0.025$. π_1: Allocate server 1 to queue 1, server 2 to queue 2. π_2: Allocate server 2 to queue 1, server 1 to queue 2. π_3: Allocate both servers to queue 1. π_4: Allocate both servers to queue 2.

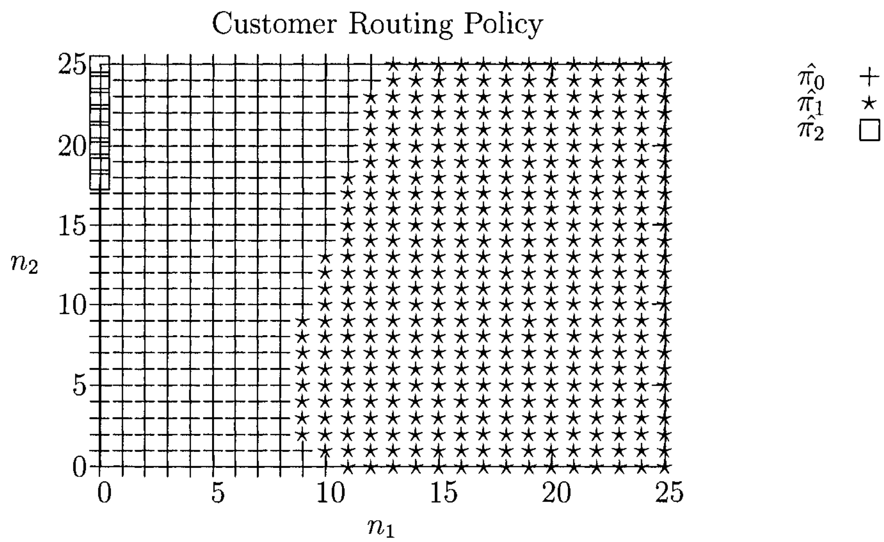


Figure 2.3: Customer routing policy for subadditive pooling rates (i). $h_1 = 10$, $h_2 = 8$, $\lambda_1 = 4$, $\lambda_2 = 5.5$, $\mu_1 = 8$, $\mu_2 = 7$, $\mu_c = 14.025$, $r = 3$, $\alpha = 0.025$. π_1: Allocate server 1 to queue 1, server 2 to queue 2. π_2: Allocate server 2 to queue 1, server 1 to queue 2. π_3: Allocate both servers to queue 1. π_4: Allocate both servers to queue 2.

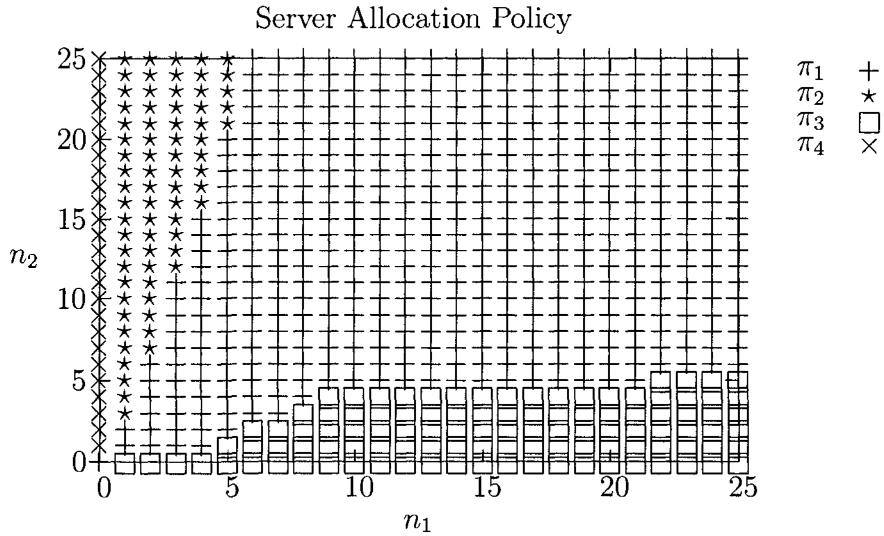


Figure 2.4: Server allocation policy for subadditive pooling rates (ii). $h_1 = 10$, $h_2 = 8$, $\lambda_1 = 4$, $\lambda_2 = 4.9$, $\mu_1 = 8$, $\mu_2 = 7$, $\mu_c = 14.025$, $r = 3$, $\alpha = 0.025$. π_1: Allocate server 1 to queue 1, server 2 to queue 2. π_2: Allocate server 2 to queue 1, server 1 to queue 2. π_3: Allocate both servers to queue 1. π_4: Allocate both servers to queue 2.

subadditive, there are times when it is optimal to route a customer from a lower cost queue to a higher cost queue (see figure 2.3). This result is somewhat counterintuitive as we consider the fact that the firm may route a customer from the lower cost queue, only to be served at a slower rate at the higher cost queue. This is an interesting insight. That is, there are times when a firm should put its best workers at lower cost facilities while putting its slower workers at high cost facilities and transfer customers from the low cost to the high cost facility. The reason for routing the customer is that if queue 1 becomes empty, then both servers will work together at queue 2, resulting in a loss of productivity. By balancing the load at both queues, the servers can work at the fastest possible rate, reducing the overall cost. The reason for placing the better worker at the lower cost queue is to 'fight fires.' As a reality check, the faster worker is only allocated to the lower cost queue when the size of the lower cost queue is much greater than the size of the higher cost queue.

Also note that the while the routing and server allocation policies appear to follow a switching curve, the curve is not monotonic in the number at each queue. We prove the nature of the routing policy switching curve for the special case of a symmetric system in the next section.

2.4.2 Special Case: Symmetric Arrivals, Service Rates, and Costs

In this section, we assume that arrival rates, service rates, and holding costs are symmetric, but that the combined service rate when both servers are pooled at the same queue is less the sum of the two service rates when separate. Specifically, $\lambda_1 = \lambda_2 = \lambda$, $\mu_1 = \mu_2 = \mu$ and that when working at the same queue, the combined rate is μ_c, $\mu_c < 2\mu$. Assume linear, symmetric holding costs such that $h(n_1, n_2) = h(n_1 + n_2)$ for some constant, $h > 0$. The optimality equation (2.2.4) becomes

$$\begin{aligned} v_{t+1}(n_1, n_2) = {} & h(n_1 + n_2) + \lambda \min[v_t(n_1 + 1, n_2), v_t(n_1, n_2 + 1) + r] \\ & + \lambda \min[v_t(n_1 + 1, n_2) + r, v_t(n_1, n_2 + 1)] \\ & + \min \begin{pmatrix} \mu v_t((n_1 - 1)^+, n_2) + \mu v_t(n_1, (n_2 - 1)^+) \\ \mu_c v_t((n_1 - 1)^+, n_2) + (2\mu - \mu_c) v_t(n_1, n_2) \\ \mu_c v_t(n_1, (n_2 - 1)^+) + (2\mu - \mu_c) v_t(n_1, n_2) \end{pmatrix} \end{aligned} \tag{2.4.6}$$

The last term is the minimum of three expressions. The first is to have one server at each queue. The second is have both servers at queue 1. The third is to have both servers at queue 2. Using a sample-path argument, Potoff, Ahn, Lewis, and Beil (2008) find the optimal server allocation policy.

Theorem 2. *(Potoff, Ahn, Lewis, and Beil 2008) The optimal server allocation policy for the decision process described in optimality equation 2.4.6 is to have the servers work at separate queues unless one queue is empty. If one queue is empty, both servers work at the nonempty queue.*

Based on this theorem, we may simplify the optimality equation. We define the following operators on functions, $f(n_1, n_2)$, for $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \mapsto \mathbb{R}$.

$$
\begin{aligned}
T_1 f(n_1, n_2) &= \min[f(n_1+1, n_2), f(n_1, n_2+1) + r] \\
T_2 f(n_1, n_2) &= \min[f(n_1+1, n_2) + r, f(n_1, n_2+1)] \\
T_3 f(n_1, n_2) &= \begin{bmatrix} \mu f(n_1-1, n_2) + \mu f(n_1, n_2-1) & \text{if } n_1, n_2 \geq 1 \\ \mu_c f(n_1-1, n_2) + (2\mu - \mu_c) f(n_1, n_2) & \text{if } n_1 \geq 1, n_2 = 0 \\ \mu_c f(n_1, n_2-1) + (2\mu - \mu_c) f(n_1, n_2) & \text{if } n_1 = 0, n_2 \geq 1 \\ 2\mu f(n_1, n_2) & \text{if } n_1 = n_2 = 0 \end{bmatrix}
\end{aligned}
$$

Using these definitions, the optimality equation (2.4.6) is rewritten.

$$
\begin{aligned}
v_{t+1}(n_1, n_2) = T v_t(n_1, n_2) = h(n_1 + n_2) + \lambda T_1 v_t(n_1, n_2) \\
+ \lambda T_2 v_t(n_1, n_2) + T_3 v_t(n_1, n_2)
\end{aligned} \tag{2.4.7}
$$

We now present the main result of this section.

Theorem 3. *For the customer routing problem as described in optimality equation 2.4.6, the optimal policy, π^*_{sub}, has the following structure.*

A. *It is never optimal to route a customer to a queue that is either longer or of the same length as the originating queue.*

B. *The optimal routing policy takes the form of an increasing switching curve such that if it is optimal to route an arriving customer from queue 1 to queue 2 in state (n_1, n_2), then it is also optimal to route an arriving customer from queue 1 to queue 2 in states (n_1+1, n_2) and (n_1+1, n_2-1). Similarly, if it is optimal to route an arriving customer from queue 2 to queue 1 in state (n_1, n_2), then it is also optimal to route an arriving customer from queue 2 to queue 1 in states (n_1, n_2+1) and (n_1-1, n_2+1).*

The proof of the following lemma is contained in the appendix.

Lemma 5. *The value function in equation 2.4.6 has the following properties under policy π^*_{sub} for all $t < \infty$:*

P1. *(a)* $\Delta v_t(n_1, n_2) = v_t(n_1 + 1, n_2) - v_t(n_1, n_2 + 1) \geq 0$ *for* $n_1 \geq n_2 \geq 0$.

(b) $\Delta v_t(n_1, n_2) = v_t(n_1 + 1, n_2) - v_t(n_1, n_2 + 1) \leq 0$ *for* $n_2 \geq n_1 \geq 0$.

P2. *(a)* $D_1\Delta v_t(n_1, n_2) = \Delta v_t(n_1 + 1, n_2) - \Delta v_t(n_1, n_2) \geq 0$ *for* $n_1 \geq n_2 \geq 0$

(b) $D_2\Delta v_t(n_1, n_2) = \Delta v_t(n_1, n_2 + 1) - \Delta v_t(n_1, n_2) \leq 0$ *for* $n_2 \geq n_1 \geq 0$ *(partial diagonal dominance).*

P3. $\Delta^{(2)} v_t(n_1, n_2) = \Delta v_t(n_1 + 1, n_2) - \Delta v_t(n_1, n_2 + 1) \geq 0$ *for* $n_1, n_2 \geq 0$.

We now show how the properties of the value function provide the structure for the optimal policy as given in Theorem 3.

Proof of Theorem 3

A. An arrival at queue 1 will be routed to queue 2 if and only if $v(n_1, n_2+1) + r \leq v(n_1 + 1, n_2)$, or $\Delta v(n_1, n_2) \geq r$. If $n_2 \geq n_1$, then $\Delta v(n_1, n_2) \leq 0$ by property **P1**(b), and it is not optimal to route an arrival from queue 1 to queue 2 when $n_2 \geq n_1$. Similarly, an arrival at queue 2 will be routed to queue 1 if and only if $v(n_1 + 1, n_2) + r \leq v(n_1, n_2 + 1)$, or $\Delta v(n_1, n_2) \leq -r$. If $n_1 \geq n_2$, then $\Delta v(n_1, n_2) \geq 0$ by property **P1**(a), and it is not optimal to route an arrival from queue 2 to queue 1 when $n_1 \geq n_2$.

B. We first note that a necessary (but not sufficient) condition for the firm to route a customer from queue 1 to queue 2 is that $n_1 > n_2$. The firm routes a customer from queue 1 to queue 2 in state (n_1, n_2) if and only if $\Delta v(n_1, n_2) = v(n_1 + 1, n_2) - v(n_1, n_2 + 1) \geq r$. Similarly, the firm routes a customer from queue 1 to queue 2 in state $(n_1 + 1, n_2)$ if and only if $\Delta v(n_1 + 1, n_2) = v(n_1 + 2, n_2) - v(n_1 + 1, n_2 + 1) \geq r$. Thus, property **P2**(a), $D_1\Delta v(n_1, n_2) \geq 0$ for

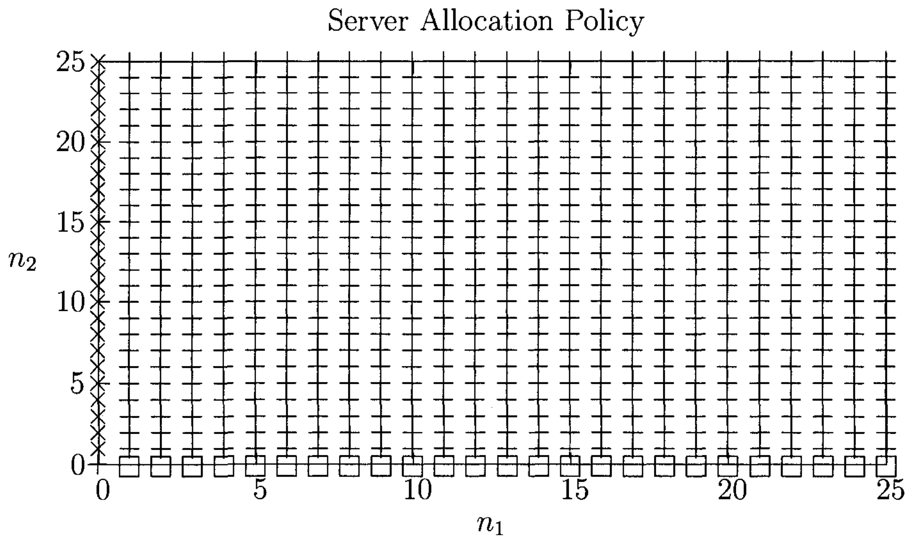


Figure 2.5: Server allocation policy for subadditive pooling rates (iii). $h_1 = h_2 = 2$, $\lambda_1 = \lambda_2 = 1.5$, $\mu_1 = \mu_2 = 2$, $\mu_c = 3.8$, $r = 0.1$, $\alpha = 0.025$. $+$: Allocate one server to each queue. $\square$: Allocate both servers to queue 1. $\times$: Allocate both servers to queue 2.

$n_1 \geq n_2 \geq 0$, is a sufficient condition to ensure that if a customer is routed from queue 1 to queue 2 in state (n_1, n_2), then a customer will also be routed from queue 1 to queue 2 in state $(n_1 + 1, n_2)$.

The firm routes an arrival from queue 1 to queue 2 in state (n_1+1, n_2-1) if and only if $\Delta v(n_1+1, n_2-1) = v(n_1+2, n_2-1) - v(n_1+1, n_2) \geq r$. Thus, property **P3**, $\Delta^{(2)} v(n_1, n_2) \geq 0$, is a sufficient condition to ensure that if a customer is routed from queue 1 to queue 2 in state (n_1, n_2), then a customer will also be routed from queue 1 to queue 2 in state $(n_1 + 1, n_2 - 1)$.

By an analogous argument, properties **P2**(b) and **P3** are sufficient conditions to ensure that if the firm routes an arrival from queue 2 to queue 1 in state (n_1, n_2), then it also routes arriving customers from queue 2 to queue 1 in states $(n_1, n_2 + 1)$ and $(n_1 - 1, n_2 + 1)$. $\square$

Examples of server allocation and routing policies are shown in figures 2.5 and 2.6,

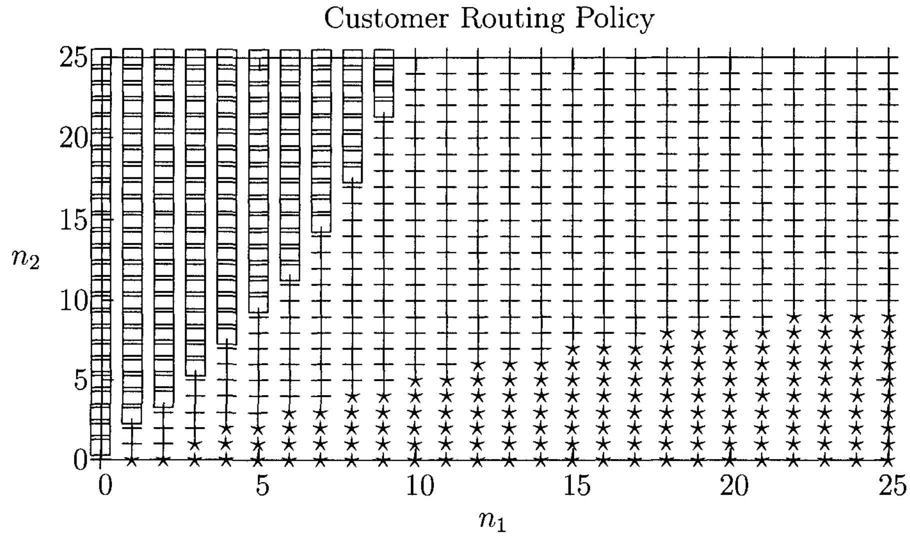


Figure 2.6: Customer routing policy for subadditive pooling rates (iii). $h_1 = h_2 = 2$, $\lambda_1 = \lambda_2 = 1.5$, $\mu_1 = \mu_2 = 2$, $\mu_c = 3.8$, $r = 0.1$, $\alpha = 0.025$. $+$: Do not route arrivals. $\star$: Route arrivals from queue 1 to queue 2. $\square$: Route arrivals from queue 2 to queue 1.

respectively. Theorem 3 tells us that the routing threshold is for queue i monotonically increasing in the number of customers at queue i. In a numerical study, however, we found that the routing threshold can be characterized more fully than what is presented in Theorem 3. In particular, the following properties, which we have seen to hold in numerical examples, would ensure that the routing threshold policy is concave and monotonic in the length of both queues.

By concave, we mean that the slope of the routing threshold is less than one. Concavity of the routing switching curve occurs if the $\Delta v(n_1, n_2) - \Delta v(n_1+1, n_2+1) \geq 0$ for $n_1 > n_2$ and $\Delta v(n_1, n_2) - \Delta v(n_1 + 1, n_2 + 1) \leq 0$ for $n_1 < n_2$. Put plainly, this condition says if we don't route an arrival from queue 1 to queue 2 in state (n_1, n_2), then we would not route an arrival from queue 1 to queue 2 in state $(n_1 + 1, n_2 + 1)$. We see this pattern very clearly in Figure 2.6.

Monotonicity of the routing switching curve in the length of both queues means that if we route from queue 1 to queue 2 in state (n_1, n_2), then we would also route from queue 1 to queue 2 in state $(n_1, n_2 - 1)$, as well as, state $(n_1 + 1, n_2)$. This monotonicity would follow immediately from $D_1 \Delta v(n_1, n_2) \geq 0$ for all n_1, n_2, not just $n_1 \geq n_2$ (as in property **P2**). To prove this, a condition which limits the growth of the value function is necessary. Specifically, we would need the condition $\mu(v(n_1, n_2 + 1) - v(n_1, n_2)) - (\mu_c - \mu)(v(n_1 + 1, n_2) - v(n_1, n_2)) \geq 0$ for $n_1 \geq n_2 \geq 0$.

Both of these properties hold in numerical examples, as no counterexamples were able to be constructed. However, it is not possible to prove them using value iteration, as the proof breaks down when evaluating boundary conditions. That is, for certain boundary conditions, it is not possible to prove (via value iteration) that the condition is either true or false. As such, we leave proof of these properties to future work.

2.5 Numerical Study

In this section, we explore the benefits of dynamic server allocation and customer routing. In order to derive meaningful results, the numerical study was conducted using the average cost criteria, as opposed to the total discounted cost criteria. We compare three cases. The first case allows only for customer routing. The second case allows for the firm to dynamically allocate the servers, but does not allow routing of arrivals. The third case allows for both customer routing and dynamic server allocation.

The main purpose of our numerical study is to find out under which circumstances each option provides the most benefit. That is, suppose a decision maker must decide which method of load balancing is best for his or her system, given that only one option (customer routing or server flexibility) is available.

Methodology

We ran a full factorial experiment on a representative sample of values for each parameter. To reduce the number of trials, we allocated the faster arrival rate to queue 1, as well as, the faster server to queue 1 in the 'routing only' case. We assumed symmetric holding costs such that $h_1 = h_2 \in \{0.6, 1.2, 1.8, 2.4\}$. Routing costs took values $r \in \{0.00, 0.25, 0.50, 1.00, 1.50\}$. We chose the service rates such that $\mu_1 + \mu_2 = 4$ with $\mu_1 \in \{2.00, 2.25, 2.50, 2.75, 3.00, 3.25, 3.50\}$ and $\mu_2 \in \{0.50, 0.75, 1.00, 1.25, 1.50, 1.75, 2.00\}$. The combined service rate was selected such that $\mu_c \in \{2.6, 3.0, 3.4, 3.8\}$. To avoid trivialities, we did not allow $\mu_c < \mu_1$. The combined arrival rate took four separate values, $\lambda_1 + \lambda_2 \in \{1.0, 2.0, 3.0, 3.25\}$. For $\lambda_1 + \lambda_2 = 1.0$, $\lambda_1 \in \{0.50, 0.65, 0.80, 0.95\}$ and $\lambda_2 \in \{0.05, 0.20, 0.35, 0.50\}$. For $\lambda_1 + \lambda_2 = 2.0$, $\lambda_1 \in \{1.00, 1.25, 1.50, 1.75\}$ and $\lambda_2 \in \{0.25, 0.50, 0.75, 1.00\}$. For $\lambda_1 + \lambda_2 = 3.0$, $\lambda_1 \in \{1.50, 1.85, 2.20, 2.55\}$ and $\lambda_2 \in \{0.45, 0.80, 1.15, 1.50\}$. For $\lambda_1 + \lambda_2 = 3.25$, $\lambda_1 \in \{1.625, 2.000, 2.375, 2.750\}$ and $\lambda_2 \in \{0.500, 0.875, 1.250, 1.625\}$. The largest value of $\lambda_1 + \lambda_2$ that we chose was 3.25. We found that for larger arrival rates, the flexible server only system could not keep up in heavy traffic and that the routing only system almost always performed better when pooling rates were subadditive.

This factorial design provided 6480 sets of parameters which were then used to find the optimal cost for each system, as well as, the costs to operate the systems which allowed either routing only or flexible servers only. In this study, the 'cost' is the gain from using the average cost formulation of our problem. In the 'routing only' system, we permanently allocate the faster server to the queue with the faster arrival rate.

We chose to observe systems with symmetric holding costs to avoid confounding our results. Higher holding costs can also be a proxy for slower service rates, as we know from queueing theory (e.g., the $c\mu$ rule). Since we really want to vary the parameters, $h_i\mu_i$, we gain more insight by maintaining symmetric holding costs in our study. Equivalently, we could have made the service rates symmetric and allowed for asymmetry in the holding costs.

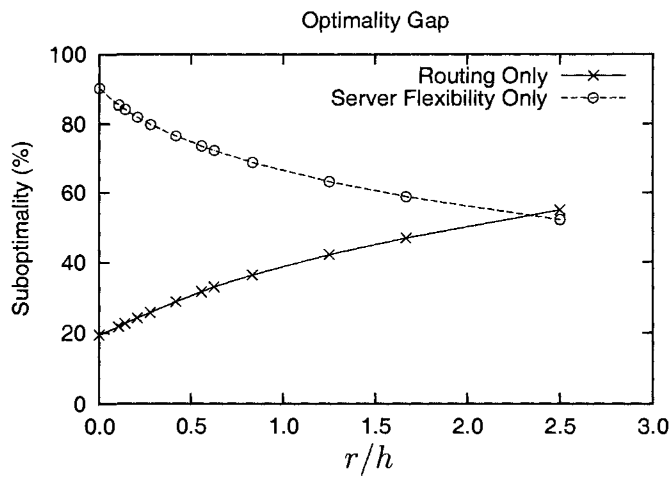


Figure 2.7: Average optimality gap as a function of routing costs relative to holding costs.

We limit our numerical study to the case of subadditive pooling rates. For additive and superadditive pooling, it is clear that the value of server flexibility will exceed the value of routing when holding costs are symmetric (see Corollary 1).

The Effect of Routing Costs. We show in Figure 2.7 that as routing costs increase (relative to the holding costs), the routing only case becomes less favorable and the flexible server only case becomes more favorable. While we did not run calculations for very large values of routing cost, it stands to reason that if the routing cost (relative to the holding cost) were above some threshold value, the optimal system would never route arrivals. This would cause the optimality gap for the flexible server only case to go to zero. Additionally, after that threshold value of routing costs is reached, the routing only system would simply be a system of two $M/M/1$ queues and the suboptimality would not be affected by an additional increase in the routing cost.

The Effect of Subadditive Pooling Rates. Figure 2.8 shows that for pooling rates which are significantly less than additive, the routing only case performs much better than the flexible server only case. The cases in which the flexible server system

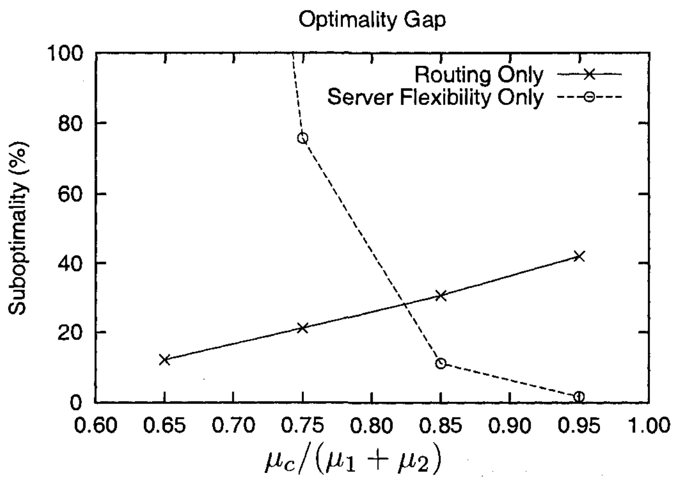


Figure 2.8: Average optimality gap as a function of combined server pooling rates.

performed the worst were typically when the combined arrival rates were high and when the arrival rates at the queues were disproportionate to service rates at the queues. For example, the worst cases had parameter values $\lambda_1 = 2.55$, $\lambda_2 = 0.45$, $\mu_1 = \mu_2 = 2.0$, $\mu_c = 2.6$. These cases boasted suboptimality of the flexible server case on the order of 3400%, while the routing only case had a supoptimality of approximately 7%. Without the ability to route customers, queue 1 would not be stable, since the single server service rate is less than the arrival rate, while the length of queue 2 would remain small. Even when the firm pooled servers at queue 1, the service rate of 2.6 is only marginally higher than the arrival rate of 2.55, and the queue should remain long. In the meantime, queue 2 is left unattended, and will also grow. By allowing routing, arrivals to queue 1 can be routed to queue 2, effectively balancing the system load.

Now, as the combined service rate grows, the flexible servers only case becomes preferred. We know from Corollary 1 that when the combined service rate is additive (or superadditive) and the holding costs are the same at both queues, the firm will never route customers. The trend in Figure 2.8 supports this result. That is, the suboptimality of of the flexible server only system tends to zero as the combined

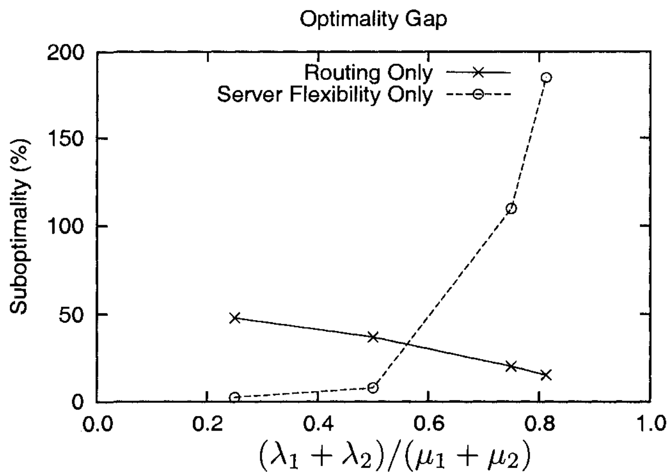
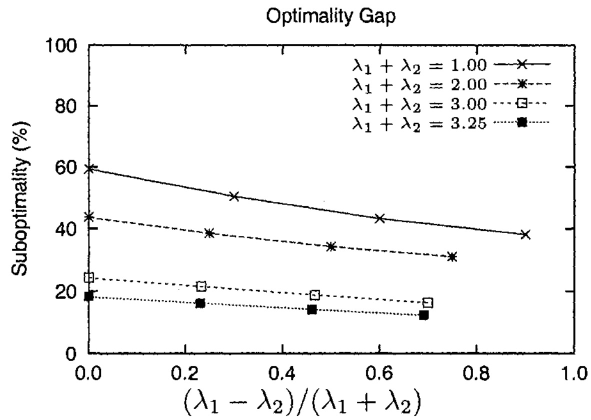


Figure 2.9: Average optimality gap as a function of traffic intensity.

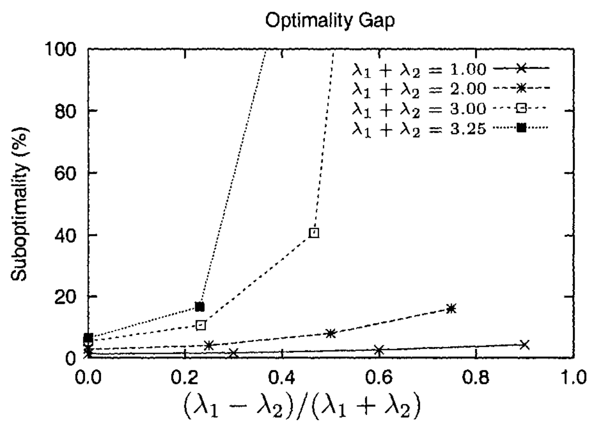
service rate approaches the additive rates of the two servers.

The Effect of Traffic Intensity. One of the most interesting findings is the effect of the traffic intensity on the suboptimality of each system. We find that the benefits of routing outweigh those of server flexibility as the traffic intensity, $(\lambda_1+\lambda_2)/(\mu_1+\mu_2)$, increases (see Figure 2.9). One of the benefits of server flexibility is that both servers will always work at a single queue when one of the queues becomes empty. This has the effect of increasing the system service rate. However, as arrival rates increase, each queue is less likely to become empty, reducing the impact of server flexibility. Furthermore, as arrival rates increase, we would prefer that the system work at the fastest possible rate, which occurs when the servers work at separate queues.

The Effect of Differences in Arrival Rates. Figures 2.10(a) shows that, for a given traffic intensity, the value of routing becomes more valuable (i.e., decreasing suboptimality) as the difference in arrival rates grows between the queues. Figure 2.10(b) shows that the opposite is true for the value of flexible servers. That is, as the difference in arrival rates grows, so does the suboptimality of the flexible server only system, especially in systems with larger arrival rates. As we can see in these

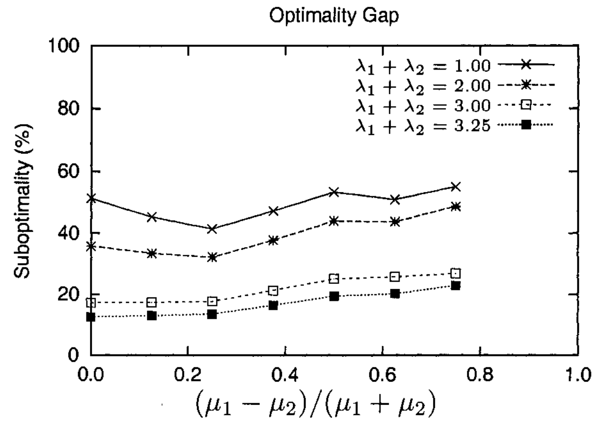


(a) Suboptimality for routing only systems.

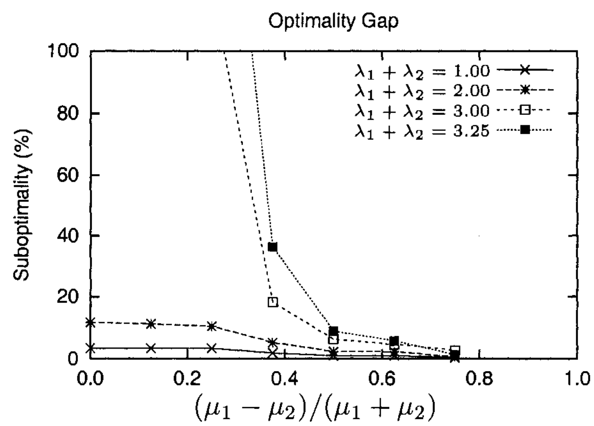


(b) Suboptimality for flexible server only systems.

Figure 2.10: Average optimality gap as a function of difference in arrival rates.



(a) Suboptimality for routing only systems.



(b) Suboptimality for flexible server only systems.

Figure 2.11: Average optimality gap as a function of difference in service rates.

figures, the effect from the difference in arrival rates, $\lambda_1 - \lambda_2)/(\lambda_1 + \lambda_2)$, is relatively small when compared to the effect of the combined arrival rates, $\lambda_1 + \lambda_2$ (or traffic intensity).

The Effect of Differences in Service Rates. For a given traffic intensity, the difference in service rates seems to have a slightly negative effect on the routing only system, but has a significant effect on the flexible server only system (see Figure 2.11(a) and 2.11(b)). That is, if there is a large difference in the service rates, the routing only system will perform worse while the flexible server only system will perform better. As we saw in the previous case, these effects are relatively small when compared to the effect of traffic intensity.

2.6 Conclusion

We have characterized the optimal policies for a system of parallel queues under varying assumption. For superadditive server pooling rates, we found that the servers never worked independently. In the case of subadditive pooling rates and a symmetric system, we found that (under certain parameter values) the servers never worked together.

Many of the properties used to prove the structure of the symmetric case of the subadditive problem could also be used to prove the structure of general subadditive service rates problem. Properties such as **P2**(a) and **P2**(b) in Lemma 5 could be used to prove a switching curve structure for both the routing and server allocation policies. While we were not able to find any counterexamples to disprove these properties in our numerical experiments, their proof remains an open problem.

Our results were provided for the infinite horizon discounted cost criteria. It can easily be shown that these results also apply to the average cost formulation of the problem.

Our numerical study highlights the fact that the combination of routing and server flexibility can provide tremendous improvement over systems which provide just one or the other. The benefits of allowing both options is amplified for systems which have significant variations in server capabilities or arrival rates at the two queues. We showed that the most significant factor in determining which option creates more value (routing or server flexibility) is the traffic intensity. For low traffic situations, server flexibility provides more benefits, while for high traffic systems, routing proves to be more beneficial.

Appendix

Proof of Lemma 1

We will prove that the value function is increasing by induction using the value iteration algorithm. For step $i = 0$, $v_0(n_1, n_2) = 0$ by assumption. It follows that for step $i = 1, v_1(n_1, n_2) = h(n_1, n_2)$, which is increasing in n_1 and n_2 by assumption. Now, assume that $v_i(n_1, n_2)$ is increasing in n_1 and n_2 for an arbitrary iteration i. Define the operators T_1, T_2, T_3, and T on functions, $f(n_1, n_2)$, for $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \mapsto \mathbb{R}$ as

$$
\begin{aligned}
T_1 f(n_1, n_2) &= \min[f(n_1 + 1, n_2), f(n_1, n_2 + 1) + r] \\
T_2 f(n_1, n_2) &= \min[f(n_1 + 1, n_2) + r, f(n_1, n_2 + 1)] \\
T_3 f(n_1, n_2) &= \min \left(\begin{array}{l}
\mu_1 f((n_1 - 1)^+, n_2) + \mu_2 f(n_1, (n_2 - 1)^+) \\
+(\Lambda - \lambda_1 - \lambda_2 - \mu_1 - \mu_2) f(n_1, n_2) \\
\\
\mu_2 f((n_1 - 1)^+, n_2) + \mu_1 f(n_1, (n_2 - 1)^+) \\
+(\Lambda - \lambda_1 - \lambda_2 - \mu_1 - \mu_2) f(n_1, n_2) \\
\\
\mu_c f((n_1 - 1)^+, n_2) + (\Lambda - \lambda_1 - \lambda_2 - \mu_c) f(n_1, n_2) \\
\\
\mu_c f(n_1, (n_2 - 1)^+) + (\Lambda - \lambda_1 - \lambda_2 - \mu_c) f(n_1, n_2)
\end{array} \right) \\
T f(n_1, n_2) &= h(n_1, n_2) + \lambda_1 T_1 f(n_1, n_2) + \lambda_2 T_2 f(n_1, n_2) + T_3 f(n_1, n_2)
\end{aligned}
$$

Since $v_{i+1}(n_1, n_2) = T v_i(n_1, n_2)$, it will be sufficient to show that $T_1 v_i(n_1, n_2), T_2 v_i(n_1, n_2)$, and $T_3 v_i(n_1, n_2)$ are increasing in n_1 and n_2.

Part 1: $T_1 v_i(n_1, n_2)$ increasing in n_1. We want to show that

$$
T_1 v_i(n_1 + 1, n_2) - T_1 v_i(n_1, n_2) = \min[v_i(n_1 + 2, n_2), v_i(n_1 + 1, n_2 + 1) + r]
$$

$$- \min[v_i(n_1+1,n_2), v_i(n_1,n_2+1)+r]$$

is nonnegative. It is sufficient to check the following two cases.

Case a: $T_1 v_i(n_1+1,n_2) = v_i(n_1+2,n_2)$.

$$\begin{aligned}
T_1 v_i(n_1+1,n_2) - T_1 v_i(n_1,n_2) &= v_i(n_1+2,n_2) - \min[v_i(n_1+1,n_2), v_i(n_1,n_2+1)+r] \\
&\geq v_i(n_1+2,n_2) - v_i(n_1+1,n_2) \\
&\geq 0
\end{aligned}$$

by the induction hypothesis.

Case b: $T_1 v_i(n_1+1,n_2) = v_i(n_1+1,n_2+1)+r$.

$$\begin{aligned}
T_1 v_i(n_1+1,n_2) - T_1 v_i(n_1,n_2) &= v_i(n_1+2,n_2) - \min[v_i(n_1+1,n_2), v_i(n_1,n_2+1)+r] \\
&\geq v_i(n_1+1,n_2+1) - v_i(n_1,n_2+1) \\
&\geq 0
\end{aligned}$$

by the induction hypothesis.

Part 2: $T_1 v_i(n_1,n_2)$ increasing in n_2. We want to show that

$$\begin{aligned}
T_1 v_i(n_1,n_2+1) - T_1 v_i(n_1,n_2) &= \min[v_i(n_1+1,n_2+1), v_i(n_1,n_2+2)+r] \\
&\quad - \min[v_i(n_1+1,n_2), v_i(n_1,n_2+1)+r]
\end{aligned}$$

is nonnegative. It is sufficient to check the following two cases.

Case a: $T_1 v_i(n_1,n_2+1) = v_i(n_1+1,n_2+1)$.

$$\begin{aligned}
T_1 v_i(n_1,n_2+1) - T_1 v_i(n_1,n_2) &= v_i(n_1+1,n_2+1) \\
&\quad - \min[v_i(n_1+1,n_2), v_i(n_1,n_2+1)+r] \\
&\geq v_i(n_1+1,n_2+1) - v_i(n_1+1,n_2)
\end{aligned}$$

$$\geq 0$$

by the induction hypothesis.

Case b: $T_1 v_i(n_1, n_2+1) = v_i(n_1, n_2+2) + r$.

$$\begin{aligned}
T_1 v_i(n_1, n_2+1) - T_1 v_i(n_1, n_2) &= v_i(n_1, n_2+2) + r - \min[v_i(n_1+1, n_2), v_i(n_1, n_2+1) + r] \\
&\geq v_i(n_1, n_2+2) - v_i(n_1, n_2+1) \\
&\geq 0
\end{aligned}$$

by the induction hypothesis.

Identical arguments prove $T_2 v_i(n_1, n_2)$ increasing in n_1 and n_2.

Part 3: $T_3 v_i(n_1, n_2)$ increasing in n_1. We want to show that

$$\begin{aligned}
&T_3 v_i(n_1+1, n_2) - T_3 v_i(n_1, n_2) \\
&= \min \left(\begin{array}{l}
\mu_1 v_i(n_1, n_2) + \mu_2 v_i(n_1+1, (n_2-1)^+) \\
+(\Lambda - \lambda_1 - \lambda_2 - \mu_1 - \mu_2) v_i(n_1+1, n_2) \\
\\
\mu_2 v_i(n_1, n_2) + \mu_1 v_i(n_1+1, (n_2-1)^+) \\
+(\Lambda - \lambda_1 - \lambda_2 - \mu_1 - \mu_2) v_i(n_1+1, n_2) \\
\\
\mu_c v_i(n_1, n_2) + (\Lambda - \lambda_1 - \lambda_2 - \mu_c) v_i(n_1+1, n_2) \\
\\
\mu_c v_i(n_1+1, (n_2-1)^+) + (\Lambda - \lambda_1 - \lambda_2 - \mu_c) v_i(n_1+1, n_2)
\end{array} \right)
\end{aligned}$$

$$
-\min \begin{pmatrix} \mu_1 v_i((n_1-1)^+, n_2) + \mu_2 v_i(n_1, (n_2-1)^+) \\ +(\Lambda - \lambda_1 - \lambda_2 - \mu_1 - \mu_2) v_i(n_1, n_2) \\ \\ \mu_2 v_i((n_1-1)^+, n_2) + \mu_1 v_i(n_1, (n_2-1)^+) \\ +(\Lambda - \lambda_1 - \lambda_2 - \mu_1 - \mu_2) v_i(n_1, n_2) \\ \\ \mu_c v_i((n_1-1)^+, n_2) + (\Lambda - \lambda_1 - \lambda_2 - \mu_c) v_i(n_1, n_2) \\ \\ \mu_c v_i(n_1, (n_2-1)^+) + (\Lambda - \lambda_1 - \lambda_2 - \mu_c) v_i(n_1, n_2) \end{pmatrix}
$$

is nonnegative. It is sufficient to check the following four cases.

Case a: $T_3 v_i(n_1+1, n_2) = \mu_1 v_i(n_1, n_2) + \mu_2 v_i(n_1+1, (n_2-1)^+) + (\Lambda - \lambda_1 - \lambda_2 - \mu_1 - \mu_2) v_i(n_1+1, n_2)$.

$$
\begin{aligned}
T_3 v_i(n_1+1, n_2) - T_3 v_i(n_1, n_2) &\geq \mu_1 (v_i(n_1, n_2) - v_i((n_1-1)^+, n_2)) \\
&\quad + \mu_2 (v_i(n_1+1, (n_2-1)^+) - v_i(n_1, (n_2-1)^+)) \\
&\quad + (\Lambda - \lambda_1 - \lambda_2 - \mu_1 - \mu_2)(v_i(n_1+1, n_2) - v_i(n_1, n_2)) \\
&\geq 0
\end{aligned}
$$

by the induction hypothesis.

Case b: $T_3 v_i(n_1+1, n_2) = \mu_2 v_i(n_1, n_2) + \mu_1 v_i(n_1+1, (n_2-1)^+) + (\Lambda - \lambda_1 - \lambda_2 - \mu_1 - \mu_2) v_i(n_1+1, n_2)$.

$$
\begin{aligned}
T_3 v_i(n_1+1, n_2) - T_3 v_i(n_1, n_2) &\geq \mu_2 (v_i(n_1, n_2) - v_i((n_1-1)^+, n_2)) \\
&\quad + \mu_1 (v_i(n_1+1, (n_2-1)^+) - v_i(n_1, (n_2-1)^+)) \\
&\quad + (\Lambda - \lambda_1 - \lambda_2 - \mu_1 - \mu_2)(v_i(n_1+1, n_2) - v_i(n_1, n_2)) \\
&\geq 0
\end{aligned}
$$

by the induction hypothesis.

Case c: $T_3 v_i(n_1+1, n_2) = \mu_c v_i((n_1-1)^+, n_2) + (\Lambda - \lambda_1 - \lambda_2 - \mu_c) v_i(n_1, n_2)$.

$$
\begin{aligned}
T_3 v_i(n_1+1, n_2) - T_3 v_i(n_1, n_2) &\geq \mu_c(v(n_1, n_2) - v((n_1-1)^+, n_2)) \\
&\quad + (\Lambda - \lambda_1 - \lambda_2 - \mu_c)(v_i(n_1+1, n_2) - v_i(n_1, n_2)) \\
&\geq 0
\end{aligned}
$$

by the induction hypothesis.

Case c: $T_3 v_i(n_1+1, n_2) = \mu_c v_i(n_1, (n_2-1)^+) + (\Lambda - \lambda_1 - \lambda_2 - \mu_c) v_i(n_1, n_2)$.

$$
\begin{aligned}
T_3 v_i(n_1+1, n_2) - T_3 v_i(n_1, n_2) &\geq \mu_c(v(n_1+1, (n_2-1)^+) - v(n_1, (n_2-1)^+)) \\
&\quad + (\Lambda - \lambda_1 - \lambda_2 - \mu_c)(v_i(n_1+1, n_2) - v_i(n_1, n_2)) \\
&\geq 0
\end{aligned}
$$

by the induction hypothesis.

A similar argument is used to show $T_3 v_i(n_1, n_2+1) - T_3 v_i(n_1, n_2) \geq 0$. Since T is a contraction mapping, $Tv_i(n_1, n_2)$ is a convergent sequence, and the limit of the sequence, $v_\infty(n_1, n_2)$, also has the property that it is increasing in n_1 and n_2. □

Proof of Lemma 4

Define the following operators on functions, $f(n_1, n_2)$, for $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \mapsto \mathbb{R}$.

$$
\begin{aligned}
\tilde{T}_1 f(n_1, n_2) &= \min[f(n_1+1, n_2), f(n_1, n_2+1) + r] \\
\tilde{T}_2 f(n_1, n_2) &= \min[f(n_1+1, n_2) + r, f(n_1, n_2+1)] \\
\tilde{T}_3 f(n_1, n_2) &= \min[f((n_1-1)^+, n_2), f(n_1, (n_2-1)^+)] \\
\tilde{T} f(n_1, n_2) &= h_1 n_1 + h_2 n_2 + \lambda_1 \tilde{T}_1 f(n_1, n_2) + \lambda_2 \tilde{T}_2 f(n_1, n_2) + \mu_c \tilde{T}_3 f(n_1, n_2)
\end{aligned}
$$

The desired properties (**M1** - **M3**) will follow by showing that for any step of the value iteration algorithm, $\tilde{T}v_i(n_1, n_2) \in V$, where the set V is defined as the set of functions, $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \mapsto \mathbb{R}$, such that that the following properties hold for all $f \in V$.

M1. $D_1 f(n_1, n_2), D_2 f(n_1, n_2) \geq 0$ for $n_1, n_2 \geq 0$ (increasing).

M2. $\Delta f(n_1, n_2) \geq 0$ for $n_1, n_2 \geq 0$.

M3. (a) $D_1 \Delta f(n_1, n_2) \geq 0$ for $n_1, n_2 \geq 0$.

(b) $D_2 \Delta f(n_1, n_2) \leq 0$ for $n_1, n_2 \geq 0$ (diagonal dominance).

Assume that $v_0(n_1, n_2) = 0$ for every state (n_1, n_2). It is immediate that $v_1(n_1, n_2) = \tilde{T}v_0(n_1, n_2) = h_1 n_1 + h_2 n_2$, and it is easy to show that $\tilde{T}v_1(n_1, n_2) \in V$ for all $n_1, n_2 \geq 0$. Assume that $v_i(n_1, n_2) \in V$ for some iteration i. Then, $\tilde{T}v_i(n_1, n_2) \in V$ by lemmas 1, 6, 7, and 8.

Since the operator $\tilde{T}$ is a contraction mapping, the sequence $\tilde{T}v_i(n_1, n_2)$ converges to the limit, $v_\infty(n_1, n_2)$. Under the L^∞ metric, the limit of any convergent sequence of functions which satisfy the properties of V will also satisfy the the properties of V, since the set V is complete. Consider a structured decision rule, π^*_{sup}, such that properties **A**, **B**, and **C** of Theorem 1 are followed. Since $\tilde{T}v_i(n_1, n_2) \in V$, the structured decision rule, π^*_{sup}, is optimal for the one stage problem with terminal cost $v_\infty(n_1, n_2)$. Existence of a structured decision rule follows from Theorem 5.1 of Porteus (1982). □

Lemma 6. *If $f \in V$ then $\Delta \tilde{T} f(n_1, n_2) \geq 0$ for $n_1, n_2 \geq 0$ (property* **M2***).*

Proof. We write $\Delta \tilde{T} f(n_1, n_2)$ as

$$\Delta \tilde{T} f(n_1, n_2) = \lambda_1 \Delta \tilde{T}_1 f(n_1, n_2) + \lambda_2 \Delta \tilde{T}_2 f(n_1, n_2) + \Delta \tilde{T}_3 f(n_1, n_2). \qquad (2.6.8)$$

It is sufficient to show each of the three terms on the righthand side to be nonnegative.

Part 1: $\Delta\tilde{T}_1 f(n_1, n_2) \geq 0$.

$$\begin{aligned} \tilde{T}_1 f(n_1+1, n_2) - \tilde{T}_1 f(n_1, n_2+1) &= \min[f(n_1+2, n_2), f(n_1+1, n_2+1)+r] \\ &\quad - \min[f(n_1+1, n_2+1), f(n_1, n_2+2)+r] \end{aligned}$$

It is sufficient to check the following two cases for nonnegativity.

Case a: $\tilde{T}_1 f(n_1+1, n_2) = f(n_1+2, n_2)$.

$$\begin{aligned} \tilde{T}_1 f(n_1+1, n_2) - \tilde{T}_1 f(n_1, n_2+1) &\geq f(n_1+2, n_2) - f(n_1+1, n_2+1) \\ &= \Delta f(n_1+1, n_2) \\ &\geq 0 \end{aligned}$$

since $f \in V$.

Case b: $\tilde{T}_1 f(n_1+1, n_2) = f(n_1+1, n_2+1) + r$.

$$\begin{aligned} \tilde{T}_1 f(n_1+1, n_2) - \tilde{T}_1 f(n_1, n_2+1) &\geq f(n_1+1, n_2+1) + r - f(n_1, n_2+2) - r \\ &= \Delta f(n_1, n_2+1) \\ &\geq 0 \end{aligned}$$

since $f \in V$.

Part 2: $\Delta\tilde{T}_2 f(n_1, n_2) \geq 0$. The proof is similar to above and is omitted.

Part 3: $\Delta\tilde{T}_3 f(n_1, n_2) \geq 0$.

$$\begin{aligned} \tilde{T}_3 f(n_1+1, n_2) - \tilde{T}_3 f(n_1, n_2+1) &= \min[f(n_1, n_2), f(n_1+1, (n_2-1)^+)] \\ &\quad - \min[f((n_1-1)^+, n_2+1), f(n_1, n_2)] \end{aligned}$$

Since $\Delta f(n_1, n_2) \geq 0$ and $f(n_1, n_2)$ increasing in n_1 and n_2, we know that $f(n_1, n_2) \leq$

$f(n_1+1,(n_2-1)^+)$ for $f \in V$. Thus,

$$\begin{aligned} \Delta\tilde{T}_3 f(n_1,n_2) &= f(n_1,n_2) - \min[f((n_1-1)^+,n_2+1), f(n_1,n_2)] \\ &\geq f(n_1,n_2) - f(n_1,n_2) = 0. \end{aligned}$$

□

Lemma 7. *If $f \in V$ then $D_1\Delta\tilde{T}f(n_1,n_2) \geq 0$ for $n_1,n_2 \geq 0$ (property* **M3**(a)*).*

Proof. We write $D_1\Delta\tilde{T}f(n_1,n_2)$ as

$$\begin{aligned} D_1\Delta\tilde{T}f(n_1,n_2) = \lambda_1 D_1\Delta\tilde{T}_1 f(n_1,n_2) + \lambda_2 D_1\Delta\tilde{T}_2 f(n_1,n_2) \\ + D_1\Delta\tilde{T}_3 f(n_1,n_2). \end{aligned} \tag{2.6.9}$$

It is sufficient to show each of the three terms on the righthand side to be nonnegative.

Part 1: $D_1\Delta\tilde{T}_1 f(n_1,n_2) \geq 0$.

$$\begin{aligned} D_1\Delta\tilde{T}_1 f(n_1,n_2) &= \Delta\tilde{T}_1 f(n_1+1,n_2) - \Delta\tilde{T}_1 f(n_1,n_2) \\ &= \tilde{T}_1 f(n_1+2,n_2) - \tilde{T}_1 f(n_1+1,n_2+1) \\ &\quad -\tilde{T}_1 f(n_1+1,n_2) + \tilde{T}_1 f(n_1,n_2+1) \\ &= \min[f(n_1+3,n_2), f(n_1+2,n_2+1)+r] \\ &\quad - \min[f(n_1+2,n_2+1), f(n_1+1,n_2+2)+r] \\ &\quad - \min[f(n_1+2,n_2), f(n_1+1,n_2+1)+r] \\ &\quad + \min[f(n_1+1,n_2+1), f(n_1,n_2+2)+r] \end{aligned}$$

It is sufficient to check the following cases for nonnegativity.

Case a: $\tilde{T}_1 f(n_1+2,n_2) = f(n_1+3,n_2)$ and $\tilde{T}_1 f(n_1,n_2+1) = f(n_1+1,n_2+1)$.

$$D_1\Delta\tilde{T}_1 f(n_1,n_2) = \tilde{T}_1 f(n_1+2,n_2) - \tilde{T}_1 f(n_1+1,n_2+1)$$

$$
\begin{aligned}
& -\tilde{T}_1 f(n_1+1,n_2)+\tilde{T}_1 f(n_1,n_2+1) \\
= \;& f(n_1+3,n_2)-\min[f(n_1+2,n_2+1),f(n_1+1,n_2+2)+r] \\
& -\min[f(n_1+2,n_2),f(n_1+1,n_2+1)+r]+f(n_1+1,n_2+1) \\
\geq \;& f(n_1+3,n_2)-f(n_1+2,n_2+1) \\
& -f(n_1+2,n_2)+f(n_1+1,n_2+1) \\
= \;& D_1\Delta f(n_1+1,n_2)\geq 0
\end{aligned}
$$

for $f \in V$.

Case b: $\tilde{T}_1 f(n_1+2,n_2) = f(n_1+3,n_2)$ and $\tilde{T}_1 f(n_1,n_2+1) = f(n_1,n_2+2)+r$. This case violates the optimal customer routing policy. Specifically, this case says that it is optimal to route a customer from queue 1 to queue 2 in state (n_1,n_2+1), but it is also optimal to keep an arriving customer at queue 1 in state (n_1+2,n_2). This violates properties **M3**(a) and **M3**(b), $D_1\Delta f(n_1,n_2)\geq 0$ and $D_2\Delta f(n_1,n_2)\leq 0$. It is not necessary to verify nonnegativity in this case.

Case c: $\tilde{T}_1 f(n_1+2,n_2) = f(n_1+2,n_2+1)+r$ and $\tilde{T}_1 f(n_1,n_2+1) = f(n_1+1,n_2+1)$.

$$
\begin{aligned}
D_1\Delta\tilde{T}_1 f(n_1,n_2) &= \tilde{T}_1 f(n_1+2,n_2)-\tilde{T}_1 f(n_1+1,n_2+1) \\
& \quad -\tilde{T}_1 f(n_1+1,n_2)+\tilde{T}_1 f(n_1,n_2+1) \\
&= f(n_1+2,n_2+1)+r-\min[f(n_1+2,n_2+1),f(n_1+1,n_2+2)+r] \\
& \quad -\min[f(n_1+2,n_2),f(n_1+1,n_2+1)+r]+f(n_1+1,n_2+1) \\
&\geq f(n_1+2,n_2+1)+r-f(n_1+2,n_2+1) \\
& \quad -f(n_1+1,n_2+1)-r+f(n_1+1,n_2+1)=0
\end{aligned}
$$

Case d: $\tilde{T}_1 f(n_1+2,n_2) = f(n_1+2,n_2+1)+r$ and $\tilde{T}_1 f(n_1,n_2+1) = f(n_1,n_2+2)+r$. $a_2-\min[b_1,b_2]-\min[c_1,c_2]+d_2 \geq a_2-b_2-c_2+d_2 = D_1\Delta f(n_1,n_2+1)\geq 0$ since

$f \in V$.

$$\begin{aligned} D_1\Delta\tilde{T}_1 f(n_1,n_2) &= \tilde{T}_1 f(n_1+2,n_2) - \tilde{T}_1 f(n_1+1,n_2+1) \\ &\quad -\tilde{T}_1 f(n_1+1,n_2) + \tilde{T}_1 f(n_1,n_2+1) \\ &= f(n_1+2,n_2+1) + r - \min[f(n_1+2,n_2+1), f(n_1+1,n_2+2)+r] \\ &\quad - \min[f(n_1+2,n_2), f(n_1+1,n_2+1)+r] + f(n_1,n_2+2) + r \\ &\geq f(n_1+2,n_2+1) + r - f(n_1+1,n_2+2) - r \\ &\quad - f(n_1+1,n_2+1) - r + f(n_1,n_2+2) + r \\ &= D_1\Delta f(n_1,n_2+1) \geq 0 \end{aligned}$$

for $f \in V$.

Part 2: $D_1\Delta\tilde{T}_2 f(n_1,n_2) \geq 0$. Since $f \in V$, it is never optimal to route a customer from queue 2 to queue 1. Therefore, $D_1\Delta\tilde{T}_2 f(n_1,n_2) = D_1\Delta f(n_1,n_2+1) \geq 0$ for $f \in V$.

Part 3: $D_1\Delta\tilde{T}_3 f(n_1,n_2) \geq 0$.

$$\begin{aligned} \Delta\tilde{T}_3 f(n_1+1,n_2) - \Delta\tilde{T}_3 f(n_1,n_2) &= \tilde{T}_3 f(n_1+2,n_2) - \tilde{T}_3 f(n_1+1,n_2+1) \\ &\quad -\tilde{T}_3 f(n_1+1,n_2) + \tilde{T}_3 f(n_1,n_2+1) \\ &= \min[f(n_1+1,n_2), f(n_1+2,(n_2-1)^+)] \\ &\quad - \min[f(n_1,n_2+1), f(n_1+1,n_2)] \\ &\quad - \min[f(n_1,n_2), f(n_1+1,(n_2-1)^+)] \\ &\quad + \min[f((n_1-1)^+,n_2+1), f(n_1,n_2)] \end{aligned}$$

Since $f \in V$, it is optimal to allocate both servers to queue 1 unless $n_1 = 0$ and $n_2 \geq 1$. Thus,

$$\Delta\tilde{T}_3 f(n_1+1,n_2) - \Delta\tilde{T}_3 f(n_1,n_2) = f(n_1+1,n_2) - f(n_1,n_2+1) - f(n_1,n_2)$$

$$+1_{\{n_1\geq 1\}}f(n_1-1,n_2+1)+1_{\{n_1=0\}}f(n_1,n_2).$$

If $n_1 = 0$, then $D_1\Delta T_3 f(n_1,n_2) = \Delta f(n_1,n_2) \geq 0$ for $f \in V$. If $n_1 \geq 1$, then $D_1\Delta T_3 f(n_1,n_2) = D_1\Delta f(n_1-1,n_2) \geq 0$ for $f \in V$. □

Lemma 8. *If $f \in V$ then $D_2\Delta\tilde{T}f(n_1,n_2) \leq 0$ for $n_1,n_2 \geq 0$ (property* **M3**(b)*).*

Proof. We write $D_2\Delta\tilde{T}f(n_1,n_2)$ as

$$\begin{aligned} D_2\Delta\tilde{T}f(n_1,n_2) &= \lambda_1 D_2\Delta\tilde{T}_1 f(n_1,n_2)+\lambda_2 D_2\Delta\tilde{T}_2 f(n_1,n_2) \\ &\quad + D_2\Delta\tilde{T}_3 f(n_1,n_2). \end{aligned} \tag{2.6.10}$$

It is sufficient to show each of the three terms on the righthand side to be negative.

Part 1: $D_2\Delta\tilde{T}_1 f(n_1,n_2) \leq 0$.

$$\begin{aligned} D_2\Delta\tilde{T}_1 f(n_1,n_2) &= \Delta\tilde{T}_1 f(n_1,n_2+1) - \Delta\tilde{T}_1 f(n_1,n_2) \\ &= \tilde{T}_1 f(n_1+1,n_2+1) - \tilde{T}_1 f(n_1,n_2+2) \\ &\quad -\tilde{T}_1 f(n_1+1,n_2) + \tilde{T}_1 f(n_1,n_2+1) \\ &= \min[f(n_1+2,n_2+1), f(n_1+1,n_2+2)+r] \\ &\quad -\min[f(n_1+1,n_2+2), f(n_1,n_2+3)+r] \\ &\quad -\min[f(n_1+2,n_2), f(n_1+1,n_2+1)+r] \\ &\quad +\min[f(n_1+1,n_2+1), f(n_1,n_2+2)+r] \end{aligned}$$

It is sufficient to check the following cases for nonnegativity.

Case a: $\tilde{T}_1 f(n_1,n_2+2) = f(n_1+1,n_2+2)$ and $\tilde{T}_1 f(n_1+1,n_2) = f(n_1+2,n_2)$.

$$\begin{aligned} D_2\Delta\tilde{T}_1 f(n_1,n_2) &= \tilde{T}_1 f(n_1+1,n_2+1) - \tilde{T}_1 f(n_1,n_2+2) \\ &\quad -\tilde{T}_1 f(n_1+1,n_2) + \tilde{T}_1 f(n_1,n_2+1) \\ &= \min[f(n_1+2,n_2+1), f(n_1+1,n_2+2)+r] - f(n_1+1,n_2+2) \end{aligned}$$

$$
\begin{aligned}
& -f(n_1+2,n_2)+\min[f(n_1+1,n_2+1),f(n_1,n_2+2)+r] \\
\leq\ & f(n_1+2,n_2+1)-f(n_1+1,n_2+2) \\
& -f(n_1+2,n_2)+f(n_1+1,n_2+1) \\
=\ & D_2\Delta f(n_1+1,n_2)\geq 0
\end{aligned}
$$

for $f\in V$.

Case b: $\tilde{T}_1 f(n_1,n_2+2)=f(n_1+1,n_2+2)$ and $\tilde{T}_1 f(n_1+1,n_2)=f(n_1+1,n_2+1)+r$.

$$
\begin{aligned}
D_2\Delta\tilde{T}_1 f(n_1,n_2) &= \tilde{T}_1 f(n_1+1,n_2+1)-\tilde{T}_1 f(n_1,n_2+2) \\
&\quad -\tilde{T}_1 f(n_1+1,n_2)+\tilde{T}_1 f(n_1,n_2+1) \\
&= \min[f(n_1+2,n_2+1),f(n_1+1,n_2+2)+r]-f(n_1+1,n_2+2) \\
&\quad -f(n_1+1,n_2+1)-r+\min[f(n_1+1,n_2+1),f(n_1,n_2+2)+r] \\
&\leq f(n_1+1,n_2+2)+r-f(n_1+1,n_2+2) \\
&\quad -f(n_1+1,n_2+1)-r+f(n_1+1,n_2+1) \\
&= 0.
\end{aligned}
$$

Case c: $\tilde{T}_1 f(n_1,n_2+2)=f(n_1,n_2+3)+r$ and $\tilde{T}_1 f(n_1+1,n_2)=f(n_1+2,n_2)$. This case violates the optimal customer routing policy. Specifically, this case says that it is optimal to route a customer from queue 1 to queue 2 in state (n_1,n_2+2), but it is also optimal to keep an arriving customer at queue 1 in state (n_1+1,n_2). This violates properties $D_1\Delta f(n_1,n_2)\geq 0$ and $D_2\Delta f(n_1,n_2)\leq 0$. It is not necessary to verify nonnegativity in this case.

Case d: $\tilde{T}_1 f(n_1,n_2+2)=f(n_1,n_2+3)+r$ and $\tilde{T}_1 f(n_1+1,n_2)=f(n_1+1,n_2+1)+r$.

$$
\begin{aligned}
D_2\Delta\tilde{T}_1 f(n_1,n_2) &= \tilde{T}_1 f(n_1+1,n_2+1)-\tilde{T}_1 f(n_1,n_2+2) \\
&\quad -\tilde{T}_1 f(n_1+1,n_2)+\tilde{T}_1 f(n_1,n_2+1)
\end{aligned}
$$

$$
\begin{aligned}
&= \min[f(n_1+2,n_2+1), f(n_1+1,n_2+2)+r] - f(n_1,n_2+3)+r \\
&\quad -f(n_1+1,n_2+1)+r+\min[f(n_1+1,n_2+1), f(n_1,n_2+2)+r] \\
&\leq f(n_1+1,n_2+2)-f(n_1,n_2+3) \\
&\quad -f(n_1+1,n_2+1)+f(n_1,n_2+2) \\
&= D_2\Delta f(n_1,n_2+1) \geq 0
\end{aligned}
$$

for $f \in V$.

Part 2: $D_2\Delta\tilde{T}_2 f(n_1,n_2) \leq 0$. Since $f \in V$, it is never optimal to route a customer from queue 2 to queue 1. Therefore, $D_2\Delta\tilde{T}_2 f(n_1,n_2) = D_2\Delta f(n_1,n_2+1) \leq 0$ for $f \in V$.

Part 3: $D_2\Delta\tilde{T}_3 f(n_1,n_2) \leq 0$.

$$
\begin{aligned}
D_2\Delta\tilde{T}_3 f(n_1,n_2) &= \Delta\tilde{T}_3 f(n_1,n_2+1) - \Delta\tilde{T}_3 f(n_1,n_2) \\
&= \tilde{T}_3 f(n_1+1,n_2+1) - \tilde{T}_3 f(n_1,n_2+2) \\
&\quad -\tilde{T}_3 f(n_1+1,n_2) + \tilde{T}_3 f(n_1,n_2+1) \\
&= \min[f(n_1,n_2+1), f(n_1+1,n_2)] \\
&\quad -\min[f((n_1-1)^+,n_2+2), f(n_1,n_2+1)] \\
&\quad -\min[f(n_1,n_2), f(n_1+1,(n_2-1)^+)] \\
&\quad +\min[f((n_1-1)^+,n_2+1), f(n_1,n_2)]
\end{aligned}
$$

Since $f \in V$, it is optimal to allocate both servers to queue 1 unless $n_1 = 0$ and $n_2 \geq 1$. Thus,

$$
\begin{aligned}
D_2\Delta\tilde{T}_3 f(n_1,n_2) &= 1_{\{n_1\geq 1\}} f(n_1-1,n_2+1) + 1_{\{n_1=0\}} f(n_1,n_2) \\
&\quad -1_{\{n_1\geq 1\}} f(n_1-1,n_2+2) - 1_{\{n_1=0\}} f(n_1,n_2+1) \\
&\quad -f(n_1,n_2)+f(n_1,n_2+1).
\end{aligned}
$$

If $n_1 = 0$, then $D_2\Delta\tilde{T}_3 f(n_1, n_2) = 0$ for $f \in V$. If $n_1 \geq 1$, then $D_2\Delta\tilde{T}_3 f(n_1, n_2) = D_2\Delta f(n_1 - 1, n_2) \leq 0$ for $f \in V$. □

Proof of Lemma 5

Properties **P1** - **P3** will follow by showing that for any step of the value iteration algorithm, $Tv_t(n_1, n_2) \in V$, where the set V is defined as the set of functions, $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \mapsto \mathbb{R}$, such that that the following properties hold for all $f \in V$. Properties **P4** - **P7** are technical conditions necessary for proving **P1** - **P3**.

P1. (a) $\Delta f(n_1, n_2) = f(n_1 + 1, n_2) - f(n_1, n_2 + 1) \geq 0$ for $n_1 \geq n_2 \geq 0$.

(b) $\Delta f(n_1, n_2) = f(n_1 + 1, n_2) - f(n_1, n_2 + 1) \leq 0$ for $n_2 \geq n_1 \geq 0$.

P2. (a) $D_1\Delta f(n_1, n_2) = \Delta f(n_1 + 1, n_2) - \Delta f(n_1, n_2) \geq 0$ for $n_1 \geq n_2 \geq 0$.

(b) $D_2\Delta f(n_1, n_2) = \Delta f(n_1, n_2 + 1) - \Delta f(n_1, n_2) \leq 0$ for $n_2 \geq n_1 \geq 0$ (partial diagonal dominance).

P3. $\Delta^{(2)} v_t(n_1, n_2) = \Delta f(n_1 + 1, n_2) - \Delta f(n_1, n_2 + 1) \geq 0$ for $n_1, n_2 \geq 0$.

P4. $D_1 f(n_1, n_2), D_2 f(n_1, n_2) \geq 0$ for all $n_1, n_2 \geq 0$ (increasing).

P5. $f(n, m) = f(m, n)$ (symmetry).

P6. $D_1^{(2)} f(n_1, n_2), D_2^{(2)} f(n_1, n_2) \geq 0$ for $n_1, n_2 \geq 0$ (convexity).

P7. $D_{12} f(n_1, n_2) \geq 0$ for $n_1, n_2 \geq 0$ (supermodularity).

Assume that $v_0(n_1, n_2) = 0$ for every state (n_1, n_2). It is immediate that $v_1(n_1, n_2) = Tv_0(n_1, n_2) = h(n_1 + n_2)$, and it is easy to show that $Tv_1(n_1, n_2) \in V$ for all $n_1, n_2 \geq 0$. Assume that $v_i(n_1, n_2) \in V$ for some iteration i. Then, $Tv_i(n_1, n_2) \in V$ by Lemmas 1, 9, 10, 11, 12, 13, and 14. Since the operator T is a contraction mapping, the sequence $Tv_i(n_1, n_2)$ converges to the limit, $v_\infty(n_1, n_2)$. Under the L^∞ metric, the limit of any convergent sequence of functions which satisfy the properties of V will also satisfy the the properties of V, since the set V is complete. Consider a structured decision rule, δ, such that properties **A** and **B** of Theorem 3 are followed. Since $Tv_i(n_1, n_2) \in V$, the structured decision rule, π^*_{sub}, is optimal for the one stage problem with terminal cost $v_\infty(n_1, n_2)$. Existence of a structured decision rule follows from Theorem 5.1 of Porteus (1982). □

Lemma 9. *For all $f \in V$,*

(i) $\Delta Tf(n_1, n_2) \geq 0$ for $n_1 \geq n_2 \geq 0$ (property **P1**(a)*).*

(ii) $\Delta Tf(n_1, n_2) \leq 0$ for $n_2 \geq n_1 \geq 0$ (property **P1**(b)*).*

Proof. We will prove $\Delta Tf(n_1, n_2) \geq 0$ for $n_1 \geq n_2 \geq 0$. The proof of (ii) is omitted since it follows from symmetry (property **P5**). By definition,

$$\begin{aligned} \Delta Tf(n_1, n_2) &= Tf(n_1 + 1, n_2) - Tf(n_1, n_2 + 1) \\ &= \lambda \Delta T_1 f(n_1, n_2) + \lambda \Delta T_2 f(n_1, n_2) + \Delta T_3 f(n_1, n_2) \end{aligned} \tag{2.6.11}$$

It follows from symmetry that $\Delta Tf(n_1, n_2) = 0$ if $n_1 = n_2$. For the remainder of the proof we assume that $n_1 \geq n_2 + 1$. It is sufficient to show that $\Delta T_1 f(n_1, n_2)$, $\Delta T_2 f(n_1, n_2)$, and $\Delta T_3 f(n_1, n_2)$ are all nonnegative.

Part 1: $\Delta T_1 f(n_1, n_2) \geq 0$.

$$\begin{aligned} \Delta T_1 f(n_1, n_2) &= T_1 f(n_1 + 1, n_2) - T_1 f(n_1, n_2 + 1) \\ &= \min[f(n_1 + 2, n_2), f(n_1 + 1, n_2 + 1) + r] \\ &\quad - \min[f(n_1 + 1, n_2 + 1), f(n_1, n_2 + 2) + r] \end{aligned}$$

It is sufficient to check the following two cases.

Case a: $T_1 f(n_1+1, n_2) = f(n_1+2, n_2)$.

$$\begin{aligned}\Delta T_1 f(n_1, n_2) &= f(n_1+2, n_2) - T_1 f(n_1, n_2+1) \\ &\geq f(n_1+2, n_2) - f(n_1+1, n_2+1) \\ &= \Delta f(n_1+1, n_2) \geq 0\end{aligned}$$

since $n_1 \geq n_2 + 1$.

Case b: $T_1 f(n_1+1, n_2) = f(n_1+1, n_2+1) + r$.

$$\begin{aligned}\Delta T_1 f(n_1, n_2) &= f(n_1+1, n_2+1) + r - T_1 f(n_1, n_2+1) \\ &\geq f(n_1+1, n_2+1) + r - f(n_1+1, n_2+1) \\ &= r \geq 0\end{aligned}$$

Part 2: $\Delta T_2 f(n_1, n_2) \geq 0$.

$$\begin{aligned}T_2 f(n_1+1, n_2) - T_2 f(n_1, n_2+1) = &\min[f(n_1+2, n_2) + r, f(n_1+1, n_2+1)] \\ &- \min[f(n_1+1, n_2+1) + r, f(n_1, n_2+2)]\end{aligned}$$

It is sufficient to check the following two cases.

Case a: $T_2 f(n_1+1, n_2) = f(n_1+2, n_2) + r$.

$$\begin{aligned}\Delta T_2 f(n_1, n_2) &= f(n_1+2, n_2) + r - T_2 f(n_1, n_2+1) \\ &\geq f(n_1+2, n_2) + r - f(n_1+1, n_2+1) - r \\ &= \Delta f(n_1+1, n_2) \geq 0\end{aligned}$$

since $n_1 \geq n_2 + 1$.

Case b: $T_2 f(n_1+1, n_2) = f(n_1+1, n_1+1)$.

$$\begin{aligned}\Delta T_2 f(n_1, n_2) &= f(n_1+1, n_2+1) - T_2 f(n_1, n_2+1) \\ &\geq f(n_1+1, n_2+1) - f(n_1, n_2+2) \\ &= \Delta f(n_1, n_2+1) \geq 0\end{aligned}$$

since $n_1 \geq n_2 + 1$.

Part 3: $\Delta T_3 f(n_1, n_2) \geq 0$. Since $n_1 > n_2 \geq 0$, we can write

$$\begin{aligned}\Delta T_3 f(n_1, n_2) &= T_3 f(n_1+1, n_2) - T_3 f(n_1, n_2+1) \\ &= \begin{bmatrix} \mu f(n_1, n_2) + \mu f(n_1+1, n_2-1) & \text{if } n_1, n_2 \geq 1 \\ \mu_c f(n_1, n_2) + (2\mu - \mu_c) f(n_1+1, n_2) & \text{if } n_2 = 0 \end{bmatrix} \\ &\quad - \mu f(n_1-1, n_2+1) - \mu f(n_1, n_2)\end{aligned}$$

It is sufficient to show that the following two cases are nonnegative.

Case a: $n_2 \geq 1$.

$$\begin{aligned}\Delta T_3 f(n_1, n_2) &= \mu(f(n_1, n_2) + f(n_1+1, n_2-1) - f(n_1-1, n_2+1) - f(n_1, n_2)) \\ &= \mu(\Delta f(n_1, n_2-1) + \Delta f(n_1-1, n_2)) \geq 0\end{aligned}$$

since $n_1 \geq n_2 + 1$.

Case b: $n_2 = 0$.

$$\begin{aligned}\Delta T_3 f(n_1, n_2) &= \mu_c f(n_1, n_2) + (2\mu - \mu_c) f(n_1+1, n_2) \\ &\quad - \mu_c f((n_1-1)^+, n_2+1) - (2\mu - \mu_c) f(n_1, n_2+1) \\ &= \mu_c (f(n_1, n_2) - f(n_1-1, n_2+1)) \\ &\quad + (2\mu - \mu_c)(f(n_1+1, n_2) - f(n_1, n_2+1)) \\ &= \mu_c \Delta f(n_1-1, n_2) + (2\mu - \mu_c) \Delta f(n_1, n_2) \geq 0\end{aligned}$$

since $n_1 \geq n_2 + 1$, and the proof is complete.

□

Lemma 10. *For all* $f \in V$,

(i) $D_1 \Delta T f(n_1, n_2) \geq 0$ *for* $n_1 \geq n_2 \geq 0$ *(property* **P2**(a)*).*

(ii) $D_2 \Delta T f(n_1, n_2) \leq 0$ *for* $n_2 \geq n_1 \geq 0$ *(property* **P2**(b)*).*

Proof. We will prove $D_1 \Delta T f(n_1, n_2) \geq 0$ for $n_1 \geq n_2 \geq 0$. The proof of (ii) is omitted since it follows from symmetry (property **P5**). It follows from elementary algebra that

$$D_1 \Delta T f(n_1, n_2) = \lambda D_1 \Delta T_1 f(n_1, n_2) + \lambda D_1 \Delta T_2 f(n_1, n_2) + D_1 \Delta T_3 f(n_1, n_2).$$

Thus, it is sufficient to show that each of the terms, $D_1 \Delta T_1 f(n_1, n_2)$, $D_1 \Delta T_2 f(n_1, n_2)$, and $D_1 \Delta T_3 f(n_1, n_2)$, are nonnegative.

Part 1: $D_1 \Delta T_1 f(n_1, n_2) \geq 0$.

$$\begin{aligned}
&\Delta T_1 v(n_1 + 1, n_2) - \Delta T_1 v(n_1, n_2) \\
&\quad = T_1 f(n_1 + 2, n_2) - T_1 f(n_1 + 1, n_2 + 1) - T_1 f(n_1 + 1, n_2) + T_1 f(n_1, n_2 + 1) \\
&\quad = \min[f(n_1 + 3, n_2), f(n_1 + 2, n_2 + 1) + r] \\
&\qquad - \min[f(n_1 + 2, n_2 + 1), f(n_1 + 1, n_2 + 2) + r] \\
&\qquad - \min[f(n_1 + 2, n_2), f(n_1 + 1, n_2 + 1) + r] \\
&\qquad + \min[f(n_1 + 1, n_2 + 1), f(n_1, n_2 + 2) + r]
\end{aligned}$$

It is sufficient to show the following cases to be non-negative.

Case	$T_1 f(n_1 + 2, n_2)$	$T_1 f(n_1, n_2 + 1)$
a	$f(n_1 + 3, n_2)$	$f(n_1 + 1, n_2 + 1)$
b	$f(n_1 + 3, n_2)$	$f(n_1, n_2 + 2) + r$

c	$f(n_1+2, n_2+1)+r$	$f(n_1+1, n_2+1)$
d	$f(n_1+2, n_2+1)+r$	$f(n_1, n_2+2)+r$

Case a: $T_1 f(n_1+2, n_2) = f(n_1+3, n_2)$ and $T_1 f(n_1, n_2+1) = f(n_1+1, n_2+1)$.

$$\begin{aligned}
&f(n_1+3, n_2) - T_1 f(n_1+1, n_2+1) - T_1 f(n_1+1, n_2) + f(n_1+1, n_2+1) \\
&\quad \geq f(n_1+3, n_2) - f(n_1+2, n_2+1) - f(n_1+2, n_2) + f(n_1+1, n_2+1) \\
&\quad = \Delta f(n_1+2, n_2) - \Delta f(n_1+1, n_2) \\
&\quad = D_1 \Delta f(n_1+1, n_2) \geq 0
\end{aligned}$$

follows from $D_1 \Delta f(n_1, n_2) \geq 0$ for $f \in V$ and $n_1 \geq n_2$.

Case b: $T_1 f(n_1+2, n_2) = f(n_1+3, n_2)$ and $T_1 f(n_1, n_2+1) = f(n_1, n_2+2) + r$.

$$\begin{aligned}
&f(n_1+3, n_2) - T_1 f(n_1+1, n_2+1) - T_1 f(n_1+1, n_2) + f(n_1, n_2+2) + r \\
&\quad \geq f(n_1+3, n_2) - f(n_1+1, n_2+2) - r - f(n_1+2, n_2) + f(n_1, n_2+2) + r \\
&\quad = \Delta f(n_1+2, n_2) - \Delta f(n_1+1, n_2) + f(n_1+2, n_2+1) - f(n_1+1, n_2+2) \\
&\qquad - f(n_1+1, n_2+1) + f(n_1, n_2+2) \\
&\quad = \Delta f(n_1+2, n_2) - \Delta f(n_1+1, n_2) + \Delta f(n_1+1, n_2+1) - \Delta f(n_1, n_2+1) \\
&\quad = D_1 \Delta f(n_1+1, n_2) + D_1 \Delta f(n_1, n_2+1) \geq 0
\end{aligned}$$

follows from $D_1 \Delta f(n_1, n_2) \geq 0$ for $f \in V$ if $n_1 \geq n_2 + 1$. If $n_1 = n_2 = n$, then $D_1 \Delta f(n_1, n_2+1) = \Delta f(n+1, n+1) - \Delta f(n, n+1) = -\Delta f(n, n+1) \geq 0$ from properties **P1**(b) and **P5**.

Case c: $T_1 f(n_1+2, n_2) = f(n_1+2, n_2+1) + r$ and $T_1 f(n_1, n_2+1) = f(n_1 +$

$1, n_2 + 1)$.

$$
\begin{aligned}
f(n_1+2,n_2)+r-T_1f(n_1+1,n_2+1)-T_1f(n_1+1,n_2)+f(n_1+1,n_2+1)& \\
\geq f(n_1+2,n_2)+r-f(n_1+2,n_2+1)-f(n_1+1,n_2+1)& \\
-r+f(n_1+1,n_2+1)& \\
=0&
\end{aligned}
$$

Case d: $T_1f(n_1+2,n_2) = f(n_1+2,n_2+1)+r$ and $T_1f(n_1,n_2+1) = f(n_1,n_2+2)+r$.

$$
\begin{aligned}
f(n_1+2,n_2)+r-T_1f(n_1+1,n_2+1)-T_1f(n_1+1,n_2)+f(n_1,n_2+2)+r& \\
\geq f(n_1+2,n_2)+r-f(n_1+1,n_2+2)-r-f(n_1+1,n_2+1)-r& \\
+f(n_1,n_2+2)+r& \\
=\Delta f(n_1+1,n_2+1)-\Delta f(n_1,n_2+1)& \\
=D_1\Delta f(n_1,n_2+1)\geq 0&
\end{aligned}
$$

follows from $D_1\Delta f(n_1,n_2) \geq 0$ for $f \in V$ for $f \in V$ if $n_1 \geq n_2+1$. If $n_1 = n_2 = n$, then $D_1\Delta f(n_1,n_2+1) = \Delta f(n+1,n+1)-\Delta f(n,n+1) = -\Delta f(n,n+1) \geq 0$ from properties **P1**(b) and **P5**.

Part 2: $D_1\Delta T_2 f(n_1,n_2) \geq 0$.

$$
\begin{aligned}
&\Delta T_2 v(n_1+1,n_2)-\Delta T_2 v(n_1,n_2) \\
&\quad = T_2f(n_1+2,n_2)-T_2f(n_1+1,n_2+1)-T_2f(n_1+1,n_2)+T_2f(n_1,n_2+1) \\
&\quad = \min[f(n_1+3,n_2)+r, f(n_1+2,n_2+1)] \\
&\qquad -\min[f(n_1+2,n_2+1)+r, f(n_1+1,n_2+2)] \\
&\qquad -\min[f(n_1+2,n_2)+r, f(n_1+1,n_2+1)] \\
&\qquad +\min[f(n_1+1,n_2+1), f(n_1,n_2+2)+r]
\end{aligned}
$$

It is sufficient to show the following cases to be non-negative.

Case	$T_2f(n_1+2,n_2)$	$T_2f(n_1,n_2+1)$
a	$f(n_1+3,n_2)+r$	$f(n_1+1,n_2+1)+r$
b	$f(n_1+3,n_2)+r$	$f(n_1,n_2+2)$
c	$f(n_1+2,n_2+1)$	$f(n_1+1,n_2+1)+r$
d	$f(n_1+2,n_2+1)$	$f(n_1,n_2+2)$

Since $n_1 \geq n_2$, cases a and b cannot occur since we never route from the shorter queue to the longer queue (property **P1**). For the same reason, case c can only occur if $n_1 = n_2$.

Case c: $T_2f(n_1+2,n_2) = f(n_1+2,n_2+1)$ and $T_2f(n_1,n_2+1) = f(n_1+1,n_2+1)+r$.

$$\begin{aligned}
&f(n_1+2,n_2+1) - T_2f(n_1+1,n_2+1) - T_2f(n_1+1,n_2) + f(n_1+1,n_2+1) + r \\
&\quad \geq f(n_1+2,n_2+1) - f(n_1+2,n_2+1) - r - f(n_1+1,n_2+1) + f(n_1+1,n_2+1) + r \\
&\quad = 0.
\end{aligned}$$

Case d: $T_2f(n_1+2,n_2) = f(n_1+2,n_2+1)$ and $T_2f(n_1,n_2+1) = f(n_1,n_2+2)$.

$$\begin{aligned}
&f(n_1+2,n_2+1) - T_2f(n_1+1,n_2+1) - T_2f(n_1+1,n_2) + f(n_1,n_2+2) \\
&\quad \geq f(n_1+2,n_2+1) - f(n_1+1,n_2+2) - f(n_1+1,n_2+1) + f(n_1,n_2+2) \\
&\quad = D_1\Delta f(n_1,n_2+1)
\end{aligned}$$

which is nonnegative for $n_1 \geq n_2+1$. If $n_1 = n_2 = n$, then $D_1\Delta f(n_1,n_2+1) = -\Delta f(n,n+1) \geq 0$ from properties **P1**(b) and **P5**.

Part 3: $D_1 \Delta T_3 f(n_1, n_2) \geq 0$.

$$
\begin{aligned}
&\Delta T_3 f(n_1+1, n_2) - \Delta T_3 f(n_1, n_2) \\
&\quad = T_3 f(n_1+2, n_2) - T_3 f(n_1+1, n_2+1) - T_3 f(n_1+1, n_2) + T_3 f(n_1, n_2+1) \\
&\quad = \begin{bmatrix} \mu f(n_1+1, n_2) + \mu f(n_1+2, n_2-1) & \text{if } n_2 \geq 1 \\ \mu_c f(n_1+1, n_2) + (2\mu - \mu_c) f(n_1+2, n_2) & \text{if } n_2 = 0 \end{bmatrix} \\
&\qquad - \mu f(n_1, n_2+1) - \mu f(n_1+1, n_2) \\
&\qquad - \begin{bmatrix} \mu f(n_1, n_2) + \mu f(n_1+1, n_2-1) & \text{if } n_2 \geq 1 \\ \mu_c f(n_1, n_2) + (2\mu - \mu_c) f(n_1+1, n_2) & \text{if } n_2 = 0 \end{bmatrix} \\
&\qquad + \begin{bmatrix} \mu f(n_1-1, n_2+1) + \mu f(n_1, n_2) & \text{if } n_1 \geq 1 \\ \mu_c f(n_1, n_2) + (2\mu - \mu_c) f(n_1, n_2+1) & \text{if } n_1 = 0 \end{bmatrix}
\end{aligned}
$$

Since we assume $n_1 \geq n_2$, we only need to verify three cases: (a) $n_1 \geq n_2 \geq 1$, (b) $n_1 > n_2 = 0$, and (c) $n_1 = n_2 = 0$.

Case a: $n_1 \geq n_2 \geq 1$.

$$
\begin{aligned}
T_3 f(n_1&+2, n_2) - T_3 f(n_1+1, n_2+1) - T_3 f(n_1+1, n_2) + T_3 f(n_1, n_2+1) \\
&= \mu f(n_1+1, n_2) + \mu f(n_1+2, n_2-1) \\
&\quad - \mu f(n_1, n_2+1) - \mu f(n_1+1, n_2) \\
&\quad - \mu f(n_1, n_2) - \mu f(n_1+1, n_2-1) \\
&\quad + \mu f(n_1-1, n_2+1) + \mu f(n_1, n_2) \\
&= \mu(\Delta f(n_1, n_2) - \Delta f(n_1-1, n_2) + \Delta f(n_1+1, n_2-1) - \Delta f(n_1, n_2-1)) \\
&= \mu(D_1 \Delta f(n_1-1, n_2) + D_1 \Delta f(n_1, n_2-1)) \geq 0
\end{aligned}
$$

follows from $D_1 \Delta f(x_1, x_2) \geq 0$ for $f \in V$ and $x_1 \geq x_2$. This condition holds if $n_1 \geq n_2 + 1$. If $n_1 = n_2 = n$, then $D_1 \Delta f(n-1, n) = -\Delta f(n, n+1) \geq 0$ from properties **P1**(b) and **P5**.

Case b: $n_1 > n_2 = 0$.

$$
\begin{aligned}
& T_3 f(n_1+2,n_2) - T_3 f(n_1+1,n_2+1) - T_3 f(n_1+1,n_2) + T_3 f(n_1,n_2+1) \\
&\quad = \mu_c f(n_1+1,n_2) + (2\mu-\mu_c) f(n_1+2,n_2) \\
&\qquad -\mu f(n_1,n_2+1) - \mu f(n_1+1,n_2) \\
&\qquad -\mu_c f(n_1,n_2) - (2\mu-\mu_c) f(n_1+1,n_2) \\
&\qquad +\mu f(n_1-1,n_2+1) + \mu f(n_1,n_2) \\
&\quad = \mu_c (f(n_1+1,n_2) - f(n_1,n_2)) + (2\mu-\mu_c)(f(n_1+2,n_2) - f(n_1+1,n_2)) \\
&\qquad -\mu(f(n_1+1,n_2) - f(n_1,n_2) + f(n_1,n_2+1) - f(n_1-1,n_2+1)) \\
&\quad = (\mu_c-\mu)(f(n_1+1,n_2) - f(n_1,n_2)) \\
&\qquad +(2\mu-\mu_c)(f(n_1+2,n_2) - f(n_1+1,n_2)) \\
&\qquad -\mu(f(n_1,n_2+1) - f(n_1-1,n_2+1)) \\
&\quad \geq (\mu_c-\mu)(f(n_1+1,n_2) - f(n_1,n_2)) \\
&\qquad +(2\mu-\mu_c)(f(n_1+1,n_2) - f(n_1,n_2)) \\
&\qquad -\mu(f(n_1,n_2+1) - f(n_1-1,n_2+1)) \\
&\quad = \mu(f(n_1+1,n_2) - f(n_1,n_2+1) - f(n_1,n_2) + f(n_1-1,n_2+1)) \\
&\quad = \mu(\Delta f(n_1,n_2) - \Delta f(n_1-1,n_2)) \\
&\quad = \mu D_1 \Delta f(n_1-1,n_2) \geq 0.
\end{aligned}
$$

The first inequality follows from $D_1 f(n_1,n_2)$ increasing in n_1 for $f \in V$ (property **P6**). The last inequality follows from $D_1 \Delta f(x_1,x_2) \geq 0$ for $f \in V$ and $x_1 \geq x_2$. This condition holds if $n_1 \geq n_2+1$. If $n_1 = n_2 = n$, then $D_1 \Delta f(n-1,n) = -\Delta f(n,n+1) \geq 0$ from properties **P1**(b) and **P5**.

Case c: $n_1 = n_2 = 0$.

$$\begin{aligned}
&T_3 f(n_1+2, n_2) - T_3 f(n_1+1, n_2+1) - T_3 f(n_1+1, n_2) + T_3 f(n_1, n_2+1) \\
&\quad = \mu_c f(1,0) + (2\mu - \mu_c) f(2,0) \\
&\qquad -\mu f(1,0) - \mu f(0,1) \\
&\qquad -\mu_c f(0,0) - (2\mu - \mu_c) f(1,0) \\
&\qquad +\mu_c f(0,0) + (2\mu - \mu_c) f(0,1) \\
&\quad \geq \mu_c f(1,0) + (2\mu - \mu_c) f(2,0) \\
&\qquad -\mu_c f(1,0) - (2\mu - \mu_c) f(1,1) \\
&\qquad -\mu_c f(0,0) - (2\mu - \mu_c) f(1,0) \\
&\qquad +\mu_c f(0,0) + (2\mu - \mu_c) f(0,1) \\
&\quad = (2\mu - \mu_c)(f(2,0) - f(1,1) - f(1,0) + f(0,1)) \\
&\quad = (2\mu - \mu_c) D_1 \Delta f(0,0) \geq 0.
\end{aligned}$$

The first inequality follows from the optimality of splitting the servers when both queues are not empty. The second inequality follows from $D_1 \Delta f(n_1, n_2) \geq 0$ for $f \in V$ and $n_1 \geq n_2 \geq 0$.

□

Lemma 11. *If $f \in V$ and $n_1, n_2 \geq 0$, then $\Delta^{(2)} T f(n_1, n_2) \geq 0$ (property* **P3***).*

Proof. We begin by rewriting $\Delta^{(2)} T f(n_1, n_2)$ as the sum of three terms,

$$\Delta^{(2)} T f(n_1, n_2) = \lambda \Delta^{(2)} T_1 f(n_1, n_2) + \lambda \Delta^{(2)} T_2 f(n_1, n_2) + \Delta^{(2)} T_3 f(n_1, n_2).$$

We will show that each of the terms, $\Delta^{(2)} T_1 f(n_1, n_2)$, $\Delta^{(2)} T_2 f(n_1, n_2)$, and $\Delta^{(2)} T_3 f(n_1, n_2)$, are nonnegative.

Part 1: $\Delta^{(2)} T_1 f(n_1, n_2) \geq 0.$

$$\begin{aligned}
\Delta^{(2)} T_1 f(n_1, n_2) &= \Delta T_1 f(n_1+1, n_2) - \Delta T_1 f(n_1, n_2+1) \\
&= T_1 f(n_1+2, n_2) - T_1 f(n_1+1, n_2+1) \\
&\quad - T_1 f(n_1+1, n_2+1) + T_1 f(n_1, n_2+2) \\
&= \min \begin{pmatrix} f(n_1+3, n_2) \\ f(n_1+2, n_2+1) + r \end{pmatrix} - \min \begin{pmatrix} f(n_1+2, n_2+1) \\ f(n_1+1, n_2+2) + r \end{pmatrix} \\
&\quad - \min \begin{pmatrix} f(n_1+2, n_2+1) \\ f(n_1+1, n_2+2) + r \end{pmatrix} + \min \begin{pmatrix} f(n_1+1, n_2+2) \\ f(n_1, n_2+3) + r \end{pmatrix}
\end{aligned}$$

It is sufficient to show the following cases to be non-negative.

Case	$T_1 f(n_1+2, n_2)$	$T_1 f(n_1, n_2+2)$
a	$f(n_1+3, n_2)$	$f(n_1+1, n_2+2)$
b	$f(n_1+2, n_2+1) + r$	$f(n_1+1, n_2+2)$
c	$f(n_1+2, n_2+1) + r$	$f(n_1, n_2+3) + r$

The case where $T_1 f(n_1+2, n_2) = f(n_1+3, n_2)$ and $T_1 f(n_1, n_2+2) = f(n_1, n_2+3) + r$ is not allowed since it implies that we route arrivals from queue 1 to queue 2 in state (n_1, n_2+2), but not in state (n_1+2, n_2). This violates the optimal routing policy for $f \in V$. Specifically, the condition $\Delta^{(2)} f(n_1, n_2) \geq 0$ is violated and it is not necessary to verify nonnegativity for this case.

Case a: $T_1 f(n_1+2, n_2) = f(n_1+3, n_2)$ and $T_1 f(n_1, n_2+2) = f(n_1+1, n_2+2)$.

$$\begin{aligned}
\Delta^{(2)} T_1 f(n_1, n_2) &= f(n_1+3, n_2) - T_1 f(n_1+1, n_2+1) \\
&\quad - T_1 f(n_1+1, n_2+1) + f(n_1+1, n_2+2) \\
&\geq f(n_1+3, n_2) - f(n_1+2, n_2+1) \\
&\quad - f(n_1+2, n_2+1) + f(n_1+1, n_2+2) \\
&= \Delta f(n_1+2, n_2) - \Delta f(n_1+1, n_2+1) \\
&= \Delta^{(2)} f(n_1+1, n_2) \geq 0
\end{aligned}$$

since $\Delta^{(2)} f(n_1, n_2) \geq 0$ for $f \in V$.

Case b: $T_1 f(n_1+2, n_2) = f(n_1+2, n_2+1) + r$ and $T_1 f(n_1, n_2+2) = f(n_1+1, n_2+2)$.

$$\begin{aligned}
\Delta^{(2)} T_1 f(n_1, n_2) &= f(n_1+2, n_2+1) + r - T_1 f(n_1+1, n_2+1) \\
&\quad - T_1 f(n_1+1, n_2+1) + f(n_1+1, n_2+2) \\
&\geq f(n_1+2, n_2+1) + r - f(n_1+2, n_2+1) \\
&\quad - f(n_1+1, n_2+2) - r + f(n_1+1, n_2+2) \\
&= 0
\end{aligned}$$

Case c: $T_1 f(n_1+2, n_2) = f(n_1+2, n_2+1) + r$ and $T_1 f(n_1, n_2+2) = f(n_1, n_2+$

$3) + r$.

$$\begin{aligned}
\Delta^{(2)}T_1 f(n_1,n_2) &= f(n_1+2,n_2+1)+r-T_1 f(n_1+1,n_2+1) \\
&\quad - T_1 f(n_1+1,n_2+1)+f(n_1,n_2+3)+r \\
&\geq f(n_1+2,n_2+1)+r-f(n_1+1,n_2+2)-r \\
&\quad - f(n_1+1,n_2+2)-r+f(n_1,n_2+3)+r \\
&= \Delta f(n_1+1,n_2+1)-\Delta f(n_1,n_2+2) \\
&= \Delta^{(2)} f(n_1,n_2+1) \geq 0
\end{aligned}$$

since $\Delta^{(2)} f(n_1,n_2) \geq 0$ for $f \in V$.

Part 2: $\Delta^{(2)}T_2 f(n_1,n_2) \geq 0$.

$$\begin{aligned}
\Delta^{(2)}T_2 f(n_1,n_2) &= \Delta T_2 f(n_1+1,n_2) - \Delta T_2 f(n_1,n_2+1) \\
&= T_2 f(n_1+2,n_2) - T_2 f(n_1+1,n_2+1) \\
&\quad -T_2 f(n_1+1,n_2+1) + T_2 f(n_1,n_2+2) \\
&= \min\begin{pmatrix} f(n_1+3,n_2)+r \\ f(n_1+2,n_2+1) \end{pmatrix} - \min\begin{pmatrix} f(n_1+2,n_2+1)+r \\ f(n_1+1,n_2+2) \end{pmatrix} \\
&\quad - \min\begin{pmatrix} f(n_1+2,n_2+1)+r \\ f(n_1+1,n_2+2) \end{pmatrix} + \min\begin{pmatrix} f(n_1+1,n_2+2)+r \\ f(n_1,n_2+3) \end{pmatrix}
\end{aligned}$$

It is sufficient to show the following cases to be non-negative.

Case	$T_2 f(n_1+2,n_2)$	$T_2 f(n_1,n_2+2)$
a	$f(n_1+3,n_2)+r$	$f(n_1+1,n_2+2)+r$
b	$f(n_1+2,n_2+1)$	$f(n_1+1,n_2+2)+r$
c	$f(n_1+2,n_2+1)$	$f(n_1,n_2+3)$

The case where $T_2 f(n_1+2,n_2) = f(n_1+3,n_2)+r$ and $T_2 f(n_1,n_2+2) = f(n_1,n_2+3)$

is not allowed since it implies that we route arrivals from queue 2 to queue 1 in state (n_1+2, n_2), but not in state (n_1, n_2+2). This violates the optimal routing policy for $f \in V$. Specifically, the condition $\Delta^{(2)} f(n_1, n_2) \geq 0$ is violated and it is not necessary to verify nonnegativity for this case.

Case a: $T_2 f(n_1+2, n_2) = f(n_1+3, n_2)+r$ and $T_2 f(n_1, n_2+2) = f(n_1+1, n_2+2)+r$.

$$
\begin{aligned}
\Delta^{(2)} T_2 f(n_1, n_2) &= f(n_1+3, n_2) + r - T_2 f(n_1+1, n_2+1) \\
&\quad - T_2 f(n_1+1, n_2+1) + f(n_1+1, n_2+2) + r \\
&\geq f(n_1+3, n_2) + r - f(n_1+2, n_2+1) - r \\
&\quad - f(n_1+2, n_2+1) - r + f(n_1+1, n_2+2) + r \\
&= \Delta f(n_1+2, n_2) - \Delta f(n_1+1, n_2+1) \\
&= \Delta^{(2)} f(n_1+1, n_2) \geq 0
\end{aligned}
$$

since $\Delta^{(2)} f(n_1, n_2) \geq 0$ for $f \in V$.

Case b: $T_2 f(n_1+2, n_2) = f(n_1+2, n_2+1)$ and $T_2 f(n_1, n_2+2) = f(n_1+1, n_2+2)+r$.

$$
\begin{aligned}
\Delta^{(2)} T_2 f(n_1, n_2) &= f(n_1+2, n_2+1) - T_2 f(n_1+1, n_2+1) \\
&\quad - T_2 f(n_1+1, n_2+1) + f(n_1+1, n_2+2) + r \\
&\geq f(n_1+2, n_2+1) - f(n_1+2, n_2+1) - r \\
&\quad - f(n_1+1, n_2+2) + f(n_1+1, n_2+2) + r \\
&= 0
\end{aligned}
$$

Case c: $T_2 f(n_1+2, n_2) = f(n_1+2, n_2+1)$ and $T_2 f(n_1, n_2+2) = f(n_1, n_2+3)$.

$$\begin{aligned}
\Delta^{(2)} T_2 f(n_1, n_2) &= f(n_1+2, n_2+1) - T_2 f(n_1+1, n_2+1) \\
&\quad - T_2 f(n_1+1, n_2+1) + f(n_1, n_2+3) \\
&\geq f(n_1+2, n_2+1) - f(n_1+1, n_2+2) \\
&\quad - f(n_1+1, n_2+2) + f(n_1, n_2+3) \\
&= \Delta f(n_1+1, n_2+1) - \Delta f(n_1, n_2+2) \\
&= \Delta^{(2)} f(n_1, n_2+1) \geq 0
\end{aligned}$$

since $\Delta^{(2)} f(n_1, n_2) \geq 0$ for $f \in V$.

Part 3: $\Delta^{(2)} T_3 f(n_1, n_2) \geq 0$.

$$\begin{aligned}
\Delta^{(2)} T_3 f(n_1, n_2) &= \Delta T_3 f(n_1+1, n_2) - \Delta T_3 f(n_1, n_2+1) \\
&= T_3 f(n_1+2, n_2) - T_3 f(n_1+1, n_2+1) \\
&\quad - T_3 f(n_1+1, n_2+1) + T_3 f(n_1, n_2+2) \\
&= \left[\begin{array}{ll} \mu f(n_1+1, n_2) + \mu f(n_1+2, n_2-1) & \text{if } n_2 \geq 1 \\ \mu_c f(n_1+1, n_2) + (2\mu - \mu_c) f(n_1+2, n_2) & \text{if } n_2 = 0 \end{array} \right] \\
&\quad - \mu f(n_1, n_2+1) - \mu f(n_1+1, n_2) \\
&\quad - \mu f(n_1, n_2+1) - \mu f(n_1+1, n_2) \\
&\quad + \min \left[\begin{array}{ll} \mu f(n_1-1, n_2+2) + \mu f(n_1, n_2+1) & \text{if } n_1 \geq 1 \\ \mu_c f(n_1, n_2+1) + (2\mu - \mu_c) f(n_1, n_2+2) & \text{if } n_1 = 0 \end{array} \right]
\end{aligned}$$

It is sufficient to show the following four cases to be non-negative.

Case a: $n_1, n_2 \geq 1$.

$$\begin{aligned}
\Delta^{(2)} T_3 f(n_1, n_2) &= \mu(f(n_1+1, n_2) + f(n_1+2, n_2-1)) - \mu(f(n_1, n_2+1) + f(n_1+1, n_2)) \\
&\quad - \mu(f(n_1, n_2+1) + f(n_1+1, n_2)) + \mu(f(n_1-1, n_2+2) + f(n_1, n_2+1))
\end{aligned}$$

$$
\begin{aligned}
&= \mu(\Delta f(n_1,n_2) - \Delta f(n_1-1,n_2+1) + \Delta f(n_1+1,n_2-1) - \Delta f(n_1,n_2)) \\
&= \mu(\Delta^{(2)} f(n_1-1,n_2) + \Delta^{(2)} f(n_1,n_2-1)) \geq 0
\end{aligned}
$$

since $\Delta^{(2)} f(n_1,n_2) \geq 0$ for $f \in V$.

Case b: $n_1 = 0$, $n_2 \geq 1$.

$$
\begin{aligned}
\Delta^{(2)} T_3 f(n_1,n_2) &= \mu(f(n_1+1,n_2) + f(n_1+2,n_2-1)) - T_3 f(n_1+1,n_2+1) \\
&\quad -T_3 f(n_1+1,n_2+1) + \mu_c f(n_1,n_2+1) + (2\mu-\mu_c) f(n_1,n_2+2) \\
&\geq \mu(f(n_1+1,n_2) + f(n_1+2,n_2-1) \\
&\quad -f(n_1,n_2+1) - f(n_1+1,n_2)) \\
&\quad -\mu_c f(n_1+1,n_2) - (2\mu-\mu_c) f(n_1+1,n_2+1) \\
&\quad +\mu_c f(n_1,n_2+1) + (2\mu-\mu_c) f(n_1,n_2+2) \\
&= -(\mu_c-\mu)\Delta f(n_1,n_2) - (2\mu-\mu_c)\Delta f(n_1,n_2+1) \\
&\quad +\mu\Delta f(n_1+1,n_2-1) \\
&\geq -(\mu_c-\mu)\Delta f(n_1,n_2) - (2\mu-\mu_c)\Delta f(n_1,n_2) \\
&\quad +\mu\Delta f(n_1+1,n_2-1) \\
&= \mu(\Delta f(n_1+1,n_2-1) - \Delta f(n_1,n_2) \\
&= \Delta^{(2)} f(n_1,n_2-1) \geq 0
\end{aligned}
$$

for $f \in V$. The first inequality follows from the optimality of splitting the servers in state (n_1+1,n_2+1): $\mu_c f(n_1+1,n_2) + (2\mu-\mu_c) f(n_1+1,n_2+1) \geq \mu(f(n_1,n_2+1) + f(n_1+1,n_2))$. The second inequality follows from $\Delta f(n_1,n_2)$ decreasing in n_2 for $n_2 > n_1$ and $f \in V$ (property **P2(b)**).

Case c: $n_1 \geq 1$, $n_2 = 0$.

$$
\Delta^{(2)} T_3 f(n_1,n_2) = \mu_c f(n_1+1,n_2) + (2\mu-\mu_c) f(n_1+2,n_2)
$$

$$
\begin{aligned}
& -T_3f(n_1+1,n_2+1)-T_3f(n_1+1,n_2+1) \\
& +\mu(f(n_1-1,n_2+2)+f(n_1,n_2+1)) \\
\geq\ & \mu_cf(n_1+1,n_2)+(2\mu-\mu_c)f(n_1+2,n_2) \\
& -\mu f(n_1,n_2+1)-\mu f(n_1+1,n_2) \\
& -\mu_cf(n_1,n_2+1)-(2\mu-\mu_c)f(n_1+1,n_2+1) \\
& +\mu(f(n_1-1,n_2+2)+f(n_1,n_2+1)) \\
=\ & (\mu_c-\mu)(f(n_1+1,n_2)-f(n_1,n_2+1)) \\
& +(2\mu-\mu_c)(f(n_1+2,n_2)-f(n_1+1,n_2+1)) \\
& -\mu(f(n_1,n_2+1)-f(n_1-1,n_2+2)) \\
\geq\ & (\mu_c-\mu)(f(n_1+1,n_2)-f(n_1,n_2+1)) \\
& +(2\mu-\mu_c)(f(n_1+1,n_2)-f(n_1,n_2+1)) \\
& -\mu(f(n_1,n_2+1)-f(n_1-1,n_2+2)) \\
=\ & \mu(f(n_1+1,n_2)-f(n_1,n_2+1) \\
& -f(n_1,n_2+1)+f(n_1-1,n_2+2)) \\
=\ & \mu(\Delta f(n_1,n_2)-\Delta f(n_1-1,n_2+1)) \\
=\ & \mu\Delta^{(2)}f(n_1-1,n_2)\geq 0
\end{aligned}
$$

for $f\in V$. The first inequality follows from the optimality of splitting the servers in state (n_1+1,n_2+1): $\mu_cf(n_1+1,n_2)+(2\mu-\mu_c)f(n_1+1,n_2+1)\geq \mu(f(n_1,n_2+1)+f(n_1+1,n_2))$. The second inequality follows from $\Delta f(n_1,n_2)$ increasing in n_1 for $n_1>n_2$ and $f\in V$ (property **P2**(a)).

Case d: $n_1=n_2=0$.

$$
\begin{aligned}
\Delta^{(2)}T_3f(n_1,n_2) &= \mu_cf(n_1+1,n_2)+(2\mu-\mu_c)f(n_1+2,n_2) \\
& -T_3f(n_1+1,n_2+1)-T_3f(n_1+1,n_2+1)
\end{aligned}
$$

$$
\begin{aligned}
& +\mu_c f(n_1, n_2+1) + (2\mu - \mu_c) f(n_1, n_2+2) \\
\geq \; & \mu_c f(n_1+1, n_2) + (2\mu - \mu_c) f(n_1+2, n_2) \\
& -\mu_c f(n_1, n_2+1) - (2\mu - \mu_c) f(n_1+1, n_2+1) \\
& -\mu_c f(n_1+1, n_2) - (2\mu - \mu_c) f(n_1+1, n_2+1) \\
& +\mu_c f(n_1, n_2+1) + (2\mu - \mu_c) f(n_1, n_2+2) \\
= \; & (2\mu - \mu_c)(\Delta f(n_1+1, n_2) - \Delta f(n_1, n_2+1)) \\
= \; & (2\mu - \mu_c)\Delta^{(2)} f(n_1, n_2) \geq 0
\end{aligned}
$$

The first inequality follows from the optimality of splitting the servers in state (n_1+1, n_2+1). The second inequality follows from $\Delta^{(2)} f(n_1, n_2) \geq 0$ for $f \in V$ and the proof is complete.

□

Lemma 12. *If $f \in V$ then $Tf(n,m) = Tf(m,n)$ (property* **P5***).*

Proof. Note that $Tf(n,m) = h(n+m) + \lambda T_1 f(n,m) + \lambda T_2 f(n,m) + T_3 f(n,m)$. First, we show that $T_1 f(n,m) + T_2 f(n,m) - T_1 f(m,n) - T_2 f(m,n) = 0$.

$$
\begin{aligned}
& T_1 v_i(n,m) + T_2 f(n,m) - T_1 f(m,n) - T_2 f(m,n) \\
& \qquad = \min \begin{pmatrix} f(n+1,m) \\ f(n,m+1)+r \end{pmatrix} + \min \begin{pmatrix} f(n+1,m)+r \\ f(n,m+1) \end{pmatrix} \\
& \qquad - \min \begin{pmatrix} f(m+1,n) \\ f(m,n+1)+r \end{pmatrix} - \min \begin{pmatrix} f(m+1,n)+r \\ f(n,n+1) \end{pmatrix} \\
& \qquad = 0
\end{aligned}
$$

since $\min[f(n+1,m), f(n,m+1)+r] = \min[f(m+1,n)+r, f(n,n+1)]$ and $\min[f(n+1,m)+r, f(n,m+1)] = \min[f(m+1,n), f(m,n+1)+r]$ by symmetry of the value function.

We now show $T_3 f(n,m) = T_3 f(m,n)$. There are two cases to check.

Case 1: $m, n \geq 1$.

$$\begin{aligned} T_3 f(n,m) - T_3 f(m,n) &= \mu(f(n-1,m) + f(n,m-1)) - \mu(f(m-1,n) + f(m,n-1)) \\ &= \mu(f(n-1,m) + f(n,m-1) - f(n,m-1) - f(n-1,m)) = 0 \end{aligned}$$

The second equality follows from symmetry for $f \in V$.

Case 2: $m \geq 1, n = 0$.

$$\begin{aligned} T_3 f(0,m) - T_3 f(m,0) &= \mu_c f(0,m-1) + (2\mu - \mu_c) f(0,m) \\ &\quad - \mu_c f(m-1,0) - (2\mu - \mu_c) f(m,0) \\ &= \mu_c (f(0,m-1) - f(0,m-1)) + (2\mu - \mu_c)(f(0,m) - f(0,m)) = 0 \end{aligned}$$

The second equality follows from symmetry, for $f \in V$.

Thus, $Tf(n,m) = Tf(m,n)$ for all $f \in V$. □

Lemma 13. *(Property* **P6***) For all $f \in V$,*

(i) $D_1^{(2)} Tf(n_1, n_2) \geq 0$ for $n_1, n_2 \geq 0$.

(ii) $D_2^{(2)} Tf(n_1, n_2) \geq 0$ for $n_1, n_2 \geq 0$.

Proof. We prove part (i). Part (ii) will follow by symmetry. Begin by rewriting $D_1^{(2)} Tf(n_1, n_2)$ as the sum of three terms.

$$D_1^{(2)} Tf(n_1, n_2) = \lambda D_1^{(2)} T_1 f(n_1, n_2) + \lambda D_1^{(2)} T_2 f(n_1, n_2) + D_1^{(2)} T_3 f(n_1, n_2)$$

We will show that each of these terms is nonnegative for all $f \in V$.

Part 1: $D_1^{(2)}T_1 f(n_1,n_2) \geq 0$.

$$\begin{aligned}
D_1^{(2)}T_1 f(n_1,n_2) &= D_1 T_1 f(n_1+1,n_2) - D_1 T_1 f(n_1,n_2) \\
&= T_1 f(n_1+2,n_2) - T_1 f(n_1+1,n_2) \\
&\quad -T_1 f(n_1+1,n_2) + T_1 f(n_1,n_2) \\
&= \min[f(n_1+3,n_2), f(n_1+2,n_2+1)+r] \\
&\quad -\min[f(n_1+2,n_2), f(n_1+1,n_2+1)+r] \\
&\quad -\min[f(n_1+2,n_2), f(n_1+1,n_2+1)+r] \\
&\quad +\min[f(n_1+1,n_2), f(n_1,n_2+1)+r]
\end{aligned}$$

It is sufficient to show the following cases to be non-negative.

Case	$T_1 f(n_1+2,n_2)$	$T_1 f(n_1,n_2)$
a	$f(n_1+3,n_2)$	$f(n_1+1,n_2)$
b	$f(n_1+2,n_2+1)+r$	$f(n_1+1,n_2)$
c	$f(n_1+2,n_2+1)+r$	$f(n_1,n_2+1)+r$

The case where $T_1 f(n_1+2,n_2) = f(n_1+3,n_2)$ and $T_1 f(n_1,n_2) = f(n_1,n_2+1)+r$ is not allowed since it implies that the firm keeps arrivals at queue 1 in state (n_1+2,n_2), but routes arrivals from queue 1 to queue 2 in state (n_1,n_2). This violates the optimal routing policy and this case is not considered.

Case a: $T_1 f(n_1+2,n_2) = f(n_1+3,n_2)$ and $T_1 f(n_1,n_2) = f(n_1+1,n_2)$.

$$\begin{aligned}
D_1^{(2)}T_1 f(n_1,n_2) &= f(n_1+3,n_2) - T_1 f(n_1+1,n_2) \\
&\quad -T_1 f(n_1+1,n_2) + f(n_1+1,n_2) \\
&\geq f(n_1+3,n_2) - f(n_1+2,n_2) \\
&\quad -f(n_1+2,n_2) + f(n_1+1,n_2)
\end{aligned}$$

$$
\begin{aligned}
&= D_1 f(n_1+2,n_2) - D_1 f(n_1+1,n_2) \\
&= D_1^{(2)} f(n_1+1,n_2) \geq 0
\end{aligned}
$$

follows from property $D_1^{(2)} f(n_1, n_2) \geq 0$ for $f \in V$.

Case b: $T_1 f(n_1+2,n_2) = f(n_1+2,n_2+1)+r$ and $T_1 f(n_1,n_2) = f(n_1+1,n_2)$.

$$
\begin{aligned}
D_1^{(2)} T_1 f(n_1,n_2) &= f(n_1+2,n_2+1) + r - T_1 f(n_1+1,n_2) \\
&\quad - T_1 f(n_1+1,n_2) + f(n_1+1,n_2) \\
&\geq f(n_1+2,n_2+1) + r - f(n_1+1,n_2+1) - r \\
&\quad - f(n_1+2,n_2) + f(n_1+1,n_2) \\
&= D_2 f(n_1+2,n_2) - D_2 f(n_1+1,n_2) \\
&= D_{12} f(n_1+1,n_2) \geq 0
\end{aligned}
$$

follows from property **P7**, $D_{12} f(n_1,n_2) \geq 0$ for $f \in V$.

Case c: $T_1 f(n_1+2,n_2) = f(n_1+2,n_2+1)+r$ and $T_1 f(n_1,n_2) = f(n_1,n_2+1)+r$.

$$
\begin{aligned}
D_1^{(2)} T_1 f(n_1,n_2) &= f(n_1+2,n_2+1) + r - T_1 f(n_1+1,n_2) \\
&\quad - T_1 f(n_1+1,n_2) + f(n_1,n_2+1) + r \\
&\geq f(n_1+2,n_2+1) + r - f(n_1+1,n_2+1) - r \\
&\quad - f(n_1+1,n_2+1) - r + f(n_1,n_2+1) + r \\
&= D_1 f(n_1+1,n_2+1) - D_1 f(n_1,n_2+1) \\
&= D_1^{(2)} f(n_1,n_2+1) \geq 0
\end{aligned}
$$

follows from property $D_1^{(2)} f(n_1, n_2) \geq 0$ for $f \in V$.

Part 2: $D_1^{(2)}T_2f(n_1,n_2) \geq 0$.

$$\begin{aligned}
D_1^{(2)}T_2f(n_1,n_2) &= D_1T_2f(n_1+1,n_2) - D_1T_2f(n_1,n_2) \\
&= T_2f(n_1+2,n_2) - T_2f(n_1+1,n_2) \\
&\quad -T_2f(n_1+1,n_2) + T_2f(n_1,n_2) \\
&= \min[f(n_1+3,n_2)+r, f(n_1+2,n_2+1)] \\
&\quad -\min[f(n_1+2,n_2)+r, f(n_1+1,n_2+1)] \\
&\quad -\min[f(n_1+2,n_2)+r, f(n_1+1,n_2+1)] \\
&\quad +\min[f(n_1+1,n_2)+r, f(n_1,n_2+1)]
\end{aligned}$$

It is sufficient to show the following cases to be non-negative.

Case	$T_2f(n_1+2,n_2)$	$T_2f(n_1,n_2)$
a	$f(n_1+3,n_2)+r$	$f(n_1+1,n_2)+r$
b	$f(n_1+2,n_2+1)$	$f(n_1+1,n_2)+r$
c	$f(n_1+2,n_2+1)$	$f(n_1,n_2+1)$

The case where $T_2f(n_1+2,n_2) = f(n_1+3,n_2)+r$ and $T_2f(n_1,n_2) = f(n_1,n_2+1)$ is not allowed since it implies that the firm routes arrivals from queue 2 to queue 1 in state (n_1+2,n_2) but keeps arrivals at queue 2 in state (n_1,n_2). This violates the optimal routing policy and this case is not considered.

Case a: $T_2f(n_1+2,n_2) = f(n_1+3,n_2)+r$ and $T_2f(n_1,n_2) = f(n_1+1,n_2)+r$.

$$\begin{aligned}
D_1^{(2)}T_2f(n_1,n_2) &= f(n_1+3,n_2)+r - T_2f(n_1+1,n_2) \\
&\quad -T_2f(n_1+1,n_2) + f(n_1+1,n_2)+r \\
&\geq f(n_1+3,n_2)+r - f(n_1+2,n_2) - r \\
&\quad -f(n_1+2,n_2) - r + f(n_1+1,n_2)+r
\end{aligned}$$

$$
\begin{aligned}
&= D_1 f(n_1+2,n_2) - D_1 f(n_1+1,n_2) \\
&= D_1^{(2)} f(n_1+1,n_2) \geq 0
\end{aligned}
$$

follows from property $D_1^{(2)} f(n_1,n_2) \geq 0$ for $f \in V$.

Case b: $T_2 f(n_1+2,n_2) = f(n_1+2,n_2+1)$ and $T_1 f(n_1,n_2) = f(n_1+1,n_2)+r$.

$$
\begin{aligned}
D_1^{(2)} T_2 f(n_1,n_2) &= f(n_1+2,n_2+1) - T_2 f(n_1+1,n_2) \\
&\quad - T_2 f(n_1+1,n_2) + f(n_1+1,n_2) + r \\
&\geq f(n_1+2,n_2+1) - f(n_1+1,n_2+1) - r \\
&\quad - f(n_1+2,n_2) + f(n_1+1,n_2) + r \\
&= D_2 f(n_1+2,n_2) - D_2 f(n_1+1,n_2) \\
&= D_{12} f(n_1+1,n_2) \geq 0
\end{aligned}
$$

follows from property **P7**, $D_{12} f(n_1,n_2) \geq 0$ for $f \in V$.

Case c: $T_2 f(n_1+2,n_2) = f(n_1+2,n_2+1)$ and $T_1 f(n_1,n_2) = f(n_1,n_2+1)$.

$$
\begin{aligned}
D_1^{(2)} T_2 f(n_1,n_2) &= f(n_1+2,n_2+1) - T_2 f(n_1+1,n_2) \\
&\quad - T_2 f(n_1+1,n_2) + f(n_1,n_2+1) \\
&\geq f(n_1+2,n_2+1) - f(n_1+1,n_2+1) \\
&\quad - f(n_1+1,n_2+1) + f(n_1,n_2+1) \\
&= D_1 f(n_1+1,n_2+1) - D_1 f(n_1,n_2+1) \\
&= D_1^{(2)} f(n_1,n_2+1) \geq 0
\end{aligned}
$$

follows from property $D_1^{(2)} f(n_1,n_2) \geq 0$ for $f \in V$.

Part 3: $D_1^{(2)}T_3 f(n_1, n_2) \geq 0$. We now show that $D_1^{(2)}T_3 f(n_1, n_2) \geq 0$.

$$
\begin{aligned}
D_1^{(2)}T_3 f(n_1,n_2) &= D_1T_3f(n_1+1,n_2) - D_1T_3f(n_1,n_2)\\
&= T_3f(n_1+2,n_2) - T_3f(n_1+1,n_2)\\
&\quad -T_3f(n_1+1,n_2) + T_3f(n_1,n_2)\\
&= \begin{bmatrix} \mu f(n_1+1,n_2) + \mu f(n_1+2,n_2-1) & \text{if } n_2 \geq 1 \\ \mu_c f(n_1+1,n_2) + (2\mu-\mu_c) f(n_1+2,n_2) & \text{if } n_2 = 0 \end{bmatrix}\\
&\quad - \begin{bmatrix} \mu f(n_1,n_2) + \mu f(n_1+1,n_2-1) & \text{if } n_2 \geq 1 \\ \mu_c f(n_1,n_2) + (2\mu-\mu_c) f(n_1+1,n_2) & \text{if } n_2 = 0 \end{bmatrix}\\
&\quad - \begin{bmatrix} \mu f(n_1,n_2) + \mu f(n_1+1,n_2-1) & \text{if } n_2 \geq 1 \\ \mu_c f(n_1,n_2) + (2\mu-\mu_c) f(n_1+1,n_2) & \text{if } n_2 = 0 \end{bmatrix}\\
&\quad + \begin{bmatrix} \mu f(n_1-1,n_2) + \mu f(n_1,n_2-1) & \text{if } n_1,n_2 \geq 1 \\ \mu_c f(n_1-1,n_2) + (2\mu-\mu_c) f(n_1,n_2) & \text{if } n_1 \geq 1, n_2 = 0 \\ \mu_c f(n_1,n_2-1) + (2\mu-\mu_c) f(n_1,n_2) & \text{if } n_1 = 0, n_2 \geq 1 \\ 2\mu f(n_1,n_2) & \text{if } n_1,n_2 = 0 \end{bmatrix}
\end{aligned}
$$

It is sufficient to show the following cases to be non-negative.

Case a: $n_1, n_2 \geq 1$.

$$
\begin{aligned}
D_1^{(2)}T_3 f(n_1,n_2) &= \mu(f(n_1+1,n_2) + f(n_1+2,n_2-1)\\
&\quad -f(n_1,n_2) - f(n_1+1,n_2-1)\\
&\quad -f(n_1,n_2) - f(n_1+1,n_2-1)\\
&\quad +f(n_1-1,n_2) + f(n_1,n_2-1))\\
&= \mu(D_1f(n_1,n_2) - D_1f(n_1-1,n_2)\\
&\quad +D_1f(n_1+1,n_2-1) - D_1f(n_1,n_2-1))\\
&= \mu(D_1^{(2)}f(n_1-1,n_2) + D_1^{(2)}f(n_1,n_2-1))
\end{aligned}
$$

$$\geq 0$$

since $D_1^{(2)} f(n_1, n_2) \geq 0$ for $f \in V$.

Case b: $n_1 \geq 1, n_2 = 0$.

$$\begin{aligned}
D_1^{(2)} T_3 f(n_1, n_2) &= \mu_c(D_1 f(n_1, n_2) - D_1 f(n_1 - 1, n_2)) \\
&\quad + (2\mu - \mu_c)(D_1 f(n_1 + 1, n_2) - D_1 f(n_1, n_2)) \\
&= \mu_c D_1^{(2)} f(n_1 - 1, n_2) + (2\mu - \mu_c) D_1^{(2)} f(n_1, n_2) \\
&\geq 0
\end{aligned}$$

since $D_1^{(2)} f(n_1, n_2) \geq 0$ for $f(n_1, n_2) \in V$.

Case c: $n_1 = 0, n_2 \geq 1$.

$$\begin{aligned}
D_1^{(2)} T_3 f(n_1, n_2) &= \mu(f(n_1 + 1, n_2) + f(n_1 + 2, n_2 - 1)) - T_3 f(n_1 + 1, n_2) \\
&\quad - T_3 f(n_1 + 1, n_2) + \mu_c f(n_1, n_2 - 1) + (2\mu - \mu_c) f(n_1, n_2) \\
&\geq \mu(f(n_1 + 1, n_2) + f(n_1 + 2, n_2 - 1)) \\
&\quad - \mu(f(n_1, n_2) + f(n_1 + 1, n_2 - 1)) \\
&\quad - \mu_c f(n_1 + 1, n_2 - 1) - (2\mu - \mu_c) f(n_1 + 1, n_2) \\
&\quad + \mu_c f(n_1, n_2 - 1) + (2\mu - \mu_c) f(n_1, n_2)) \\
&= \mu(f(n_1 + 1, n_2) + f(n_1 + 2, n_2 - 1) \\
&\quad - f(n_1, n_2) - f(n_1 + 1, n_2 - 1)) \\
&\quad - \mu_c(f(n_1 + 1, n_2 - 1) - f(n_1, n_2 - 1)) \\
&\quad - (2\mu - \mu_c)(f(n_1 + 1, n_2) - f(n_1, n_2)) \\
&= \mu(f(n_1 + 1, n_2) + f(n_1 + 2, n_2 - 1) \\
&\quad - f(n_1, n_2) - f(n_1 + 1, n_2 - 1)
\end{aligned}$$

$$
\begin{aligned}
& -f(n_1+1,n_2-1)+f(n_1,n_2-1)) \\
& -(\mu_c-\mu)(f(n_1+1,n_2-1)-f(n_1,n_2-1)) \\
& -(2\mu-\mu_c)(f(n_1+1,n_2)-f(n_1,n_2)) \\
\geq\ & \mu(f(n_1+1,n_2)+f(n_1+2,n_2-1) \\
& -f(n_1,n_2)-f(n_1+1,n_2-1) \\
& -f(n_1+1,n_2-1)+f(n_1,n_2-1)) \\
& -(\mu_c-\mu)(f(n_1+1,n_2)-f(n_1,n_2)) \\
& -(2\mu-\mu_c)(f(n_1+1,n_2)-f(n_1,n_2)) \\
=\ & \mu(f(n_1+2,n_2-1)-f(n_1+1,n_2-1) \\
& -f(n_1+1,n_2-1)+f(n_1,n_2-1)) \\
=\ & \mu(D_1f(n_1+1,n_2-1)-D_1f(n_1,n_2-1) \\
=\ & \mu D_1^{(2)}f(n_1,n_2-1)\geq 0.
\end{aligned}
$$

The first inequality follows from the optimality of splitting the servers in state (n_1+1,n_2). The second inequality follows from $D_1f(n_1,n_2)$ increasing in n_2 for $f(n_1,n_2)\in V$ (property **P7**). The last inequality follows from $D_1^{(2)}f(n_1,n_2)\geq 0$ for $f(n_1,n_2)\in V$.

Case d: $n_1=n_2=0$. This implies that $T_3f(n_1+2,n_2)=\mu_cf(1,0)+(2\mu-\mu_c)f(2,0)$, $T_3f(n_1+1,n_2)=\mu_cf(0,0)+(2\mu-\mu_c)f(1,0)$ and $T_3f(n_1,n_2)=\mu_cf(0,0)+(2\mu-\mu_c)f(0,0)$.

$$
\begin{aligned}
D_1^{(2)}T_3f(n_1,n_2) &= \mu_c(f(1,0)-f(0,0)-f(0,0)+f(0,0)) \\
&\quad +(2\mu-\mu_c)(f(2,0)-f(1,0)-f(1,0)+f(0,0)) \\
&= \mu_c(f(1,0)-f(0,0))+(2\mu-\mu_c)(D_1f(1,0)-D_1f(0,0)) \\
&= \mu_c(f(1,0)-f(0,0))+(2\mu-\mu_c)D_1^{(2)}f(0,0)\geq 0
\end{aligned}
$$

since $f(n_1, n_2)$ is increasing in n_1 and $D_1^{(2)} f(n_1, n_2) \geq 0$ for $f \in V$ and the proof is complete.

□

Lemma 14. *If $f \in V$, then $D_{12}Tf(n_1, n_2) \geq 0$ for $n_1, n_2 \geq 0$ (property* **P7***).*

Proof. **Part 1:** $D_{12}T_1 f(n_1, n_2) \geq 0$

$$\begin{aligned} D_{12}T_1 f(n_1, n_2) &= D_1 T_1 f(n_1, n_2 + 1) - D_1 T_1 f(n_1, n_2) \\ &= T_1 f(n_1 + 1, n_2 + 1) - T_1 f(n_1, n_2 + 1) \\ &\quad - T_1 f(n_1 + 1, n_2) + T_1 f(n_1, n_2) \\ &= \min[f(n_1 + 2, n_2 + 1), f(n_1 + 1, n_2 + 2) + r] \\ &\quad - \min[f(n_1 + 1, n_2 + 1), f(n_1, n_2 + 2) + r] \\ &\quad - \min[f(n_1 + 2, n_2), f(n_1 + 1, n_2 + 1) + r] \\ &\quad + \min[f(n_1 + 1, n_2), f(n_1, n_2 + 1) + r] \end{aligned}$$

It is sufficient to verify the following four cases for nonnegativity.

Case	$T_1 f(n_1 + 1, n_2 + 1)$	$T_1 f(n_1, n_2)$
a	$f(n_1 + 2, n_2 + 1)$	$f(n_1 + 1, n_2)$
b	$f(n_1 + 2, n_2 + 1)$	$f(n_1, n_2 + 1) + r$
c	$f(n_1 + 1, n_2 + 2) + r$	$f(n_1 + 1, n_2)$
d	$f(n_1 + 1, n_2 + 2) + r$	$f(n_1, n_2 + 1) + r$

Case a: $T_1 f(n_1 + 1, n_2 + 1) = f(n_1 + 2, n_2 + 1)$ and $T_1 f(n_1, n_2) = f(n_1 + 1, n_2)$.

$$\begin{aligned} D_{12}T_1 f(n_1, n_2) &= f(n_1 + 2, n_2 + 1) - T_1 f(n_1, n_2 + 1) \\ &\quad - T_1 f(n_1 + 1, n_2) + f(n_1 + 1, n_2) \\ &\geq f(n_1 + 2, n_2 + 1) - f(n_1 + 1, n_2 + 1) \end{aligned}$$

$$
\begin{aligned}
&-f(n_1+2, n_2) + f(n_1+1, n_2) \\
= \ & D_1 f(n_1+1, n_2+1) - D_1 f(n_1+1, n_2) \\
= \ & D_{12} f(n_1+1, n_2) \geq 0
\end{aligned}
$$

follows from property $D_{12}f(n_1, n_2) \geq 0$ for $f \in V$.

Case b: $T_1 f(n_1{+}1, n_2{+}1) = f(n_1{+}2, n_2{+}1)$ and $T_1 f(n_1, n_2) = f(n_1, n_2{+}1){+}r$.

$$
\begin{aligned}
D_{12}T_1 f(n_1, n_2) &= f(n_1+2, n_2+1) - T_1 f(n_1, n_2+1) \\
&\quad -T_1 f(n_1+1, n_2) + f(n_1, n_2+1) + r \\
&\geq f(n_1+2, n_2+1) - f(n_1+1, n_2+1) \\
&\quad -f(n_1+1, n_2+1) + f(n_1, n_2+1) \\
&= D_1 v(n_1+1, n_2+1) - D_1 v(n_1, n_2+1) \\
&= D_1^{(2)} v(n_1, n_2+1) \geq 0
\end{aligned}
$$

follows from property **P6**, $D_1^{(2)} f(n_1, n_2) \geq 0$ for $f \in V$.

Case c: $T_1 f(n_1{+}1, n_2{+}1) = f(n_1{+}1, n_2{+}2){+}r$ and $T_1 f(n_1, n_2) = f(n_1{+}1, n_2)$.

$$
\begin{aligned}
D_{12}T_1 f(n_1, n_2) &= f(n_1+1, n_2+2) + r - T_1 f(n_1, n_2+1) \\
&\quad -T_1 f(n_1+1, n_2) + f(n_1+1, n_2) \\
&\geq f(n_1+1, n_2+2) - f(n_1+1, n_2+1) \\
&\quad -f(n_1+1, n_2+1) + f(n_1+1, n_2) \\
&= D_2 f(n_1+1, n_2+1) - D_2 f(n_1+1, n_2) \\
&= D_2^{(2)} f(n_1+1, n_2) \geq 0
\end{aligned}
$$

follows from property $D_2^{(2)} f(n_1, n_2) \geq 0$ for $f \in V$.

Case d: $T_1 f(n_1+1, n_2+1) = f(n_1+1, n_2+2)+r$ and $T_1 f(n_1, n_2) = f(n_1, n_2+1)+r$.

$$\begin{aligned}
D_{12}T_1 f(n_1,n_2) &= f(n_1+1,n_2+2)+r-T_1 f(n_1,n_2+1) \\
&\quad -T_1 f(n_1+1,n_2)+f(n_1,n_2+1)+r \\
&\geq f(n_1+1,n_2+2)-f(n_1,n_2+2) \\
&\quad -f(n_1+1,n_2+1)+f(n_1,n_2+1) \\
&= D_1 f(n_1,n_2+2)-D_1 f(n_1,n_2+1) \\
&= D_{12} f(n_1,n_2+1) \geq 0
\end{aligned}$$

follows from property $D_{12}f(n_1,n_2) \geq 0$ for $f \in V$.

Part 2: $D_{12}T_2 f(n_1, n_2) \geq 0$.

$$\begin{aligned}
D_{12}T_2 f(n_1,n_2) &= D_1T_2 f(n_1,n_2+1)-D_1T_2 f(n_1,n_2) \\
&= T_2 f(n_1+1,n_2+1)-T_2 f(n_1,n_2+1) \\
&\quad -T_2 f(n_1+1,n_2)+T_2 f(n_1,n_2) \\
&= \min[f(n_1+2,n_2+1)+r, f(n_1+1,n_2+2)] \\
&\quad -\min[f(n_1+1,n_2+1)+r, f(n_1,n_2+2)] \\
&\quad -\min[f(n_1+2,n_2)+r, f(n_1+1,n_2+1)] \\
&\quad +\min[f(n_1+1,n_2)+r, f(n_1,n_2+1)]
\end{aligned}$$

It is sufficient to verify the following four cases for nonnegativity.

Case	$T_2 f(n_1+1,n_2+1)$	$T_2 f(n_1,n_2)$
a	$f(n_1+2,n_2+1)+r$	$f(n_1+1,n_2)+r$
b	$f(n_1+2,n_2+1)+r$	$f(n_1,n_2+1)$
c	$f(n_1+1,n_2+2)$	$f(n_1+1,n_2)+r$

d	$f(n_1+1, n_2+2)$	$f(n_1, n_2+1)$

Case a: $T_2 f(n_1+1, n_2+1) = f(n_1+2, n_2+1) + r$ and $T_2 f(n_1, n_2) = f(n_1 + 1, n_2) + r$.

$$\begin{aligned}
D_{12} T_2 f(n_1, n_2) &= f(n_1+2, n_2+1) - T_2 f(n_1, n_2+1) \\
&\quad - T_2 f(n_1+1, n_2) + f(n_1+1, n_2) \\
&\geq f(n_1+2, n_2+1) + r - f(n_1+1, n_2+1) - r \\
&\quad - f(n_1+2, n_2) - r + f(n_1+1, n_2) + r \\
&= D_1 f(n_1+1, n_2+1) - D_1 f(n_1+1, n_2) \\
&= D_{12} f(n_1+1, n_2) \geq 0
\end{aligned}$$

follows from property $D_{12} f(n_1, n_2) \geq 0$ for $f \in V$.

Case b: $T_2 f(n_1+1, n_2+1) = f(n_1+2, n_2+1) + r$ and $T_2 f(n_1, n_2) = f(n_1, n_2+1)$.

$$\begin{aligned}
D_{12} T_2 f(n_1, n_2) &= f(n_1+2, n_2+1) + r - T_2 f(n_1, n_2+1) \\
&\quad - T_2 f(n_1+1, n_2) + f(n_1, n_2+1) \\
&\geq f(n_1+2, n_2+1) + r - f(n_1+1, n_2+1) - r \\
&\quad - f(n_1+1, n_2+1) + f(n_1, n_2+1) \\
&= D_1 v(n_1+1, n_2+1) - D_1 v(n_1, n_2+1) \\
&= D_1^{(2)} v(n_1, n_2+1) \geq 0
\end{aligned}$$

follows from property **P6**, $D_1^{(2)} f(n_1, n_2) \geq 0$ for $f \in V$.

Case c: $T_2 f(n_1+1, n_2+1) = f(n_1+1, n_2+2)$ and $T_2 f(n_1, n_2) = f(n_1+1, n_2) + r$.

$$D_{12} T_2 f(n_1, n_2) = f(n_1+1, n_2+2) - T_2 f(n_1, n_2+1)$$

$$
\begin{aligned}
& -T_2 f(n_1+1,n_2) + f(n_1+1,n_2) + r \\
\geq\ & f(n_1+1,n_2+2) - f(n_1+1,n_2+1) - r \\
& -f(n_1+1,n_2+1) + f(n_1+1,n_2) + r \\
=\ & D_2 f(n_1+1,n_2+1) - D_2 f(n_1+1,n_2) \\
=\ & D_2^{(2)} f(n_1+1,n_2) \geq 0
\end{aligned}
$$

follows from property $D_2^{(2)} f(n_1,n_2) \geq 0$ for $f \in V$.

Case d: $T_1 f(n_1+1,n_2+1) = f(n_1+1,n_2+2)$ and $T_1 f(n_1,n_2) = f(n_1,n_2+1)$.

$$
\begin{aligned}
D_{12} T_2 f(n_1,n_2) &= f(n_1+1,n_2+2) + r - T_2 f(n_1,n_2+1) \\
&\quad -T_2 f(n_1+1,n_2) + f(n_1,n_2+1) + r \\
&\geq f(n_1+1,n_2+2) - f(n_1,n_2+2) \\
&\quad -f(n_1+1,n_2+1) + f(n_1,n_2+1) \\
&= D_1 f(n_1,n_2+2) - D_1 f(n_1,n_2+1) \\
&= D_{12} f(n_1,n_2+1) \geq 0
\end{aligned}
$$

follows from property $D_{12} f(n_1,n_2) \geq 0$ for $f \in V$.

Part 3: $D_{12} T_3 f(n_1,n_2) \geq 0$.

$$
\begin{aligned}
D_{12} T_3 f(n_1,n_2) &= D_1 T_3 f(n_1,n_2+1) - D_1 T_3 f(n_1,n_2) \\
&= T_3 f(n_1+1,n_2+1) - T_3 f(n_1,n_2+1) \\
&\quad -T_3 f(n_1+1,n_2) + T_3 f(n_1,n_2) \\
&= \mu f(n_1,n_2+1) + \mu f(n_1+1,n_2) \\
&\quad - \begin{bmatrix} \mu f(n_1-1,n_2+1) + \mu f(n_1,n_2) & \text{if } n_1 \geq 1 \\ \mu_c f(n_1,n_2) + (2\mu-\mu_c) f(n_1,n_2+1) & \text{if } n_1 = 0 \end{bmatrix}
\end{aligned}
$$

$$
-\left[\begin{array}{ll}
\mu f(n_1,n_2)+\mu f(n_1+1,n_2-1) & \text{if } n_2\geq 1\\
\mu_c f(n_1,n_2)+(2\mu-\mu_c)f(n_1+1,n_2) & \text{if } n_2=0
\end{array}\right]
$$

$$
+\left[\begin{array}{ll}
\mu f(n_1-1,n_2)+\mu f(n_1,n_2-1) & \text{if } n_1,n_2\geq 1\\
\mu_c f(n_1-1,n_2)+(2\mu-\mu_c)f(n_1,n_2) & \text{if } n_1\geq 1, n_2=0\\
\mu_c f(n_1,n_2-1)+(2\mu-\mu_c)f(n_1,n_2) & \text{if } n_1=0, n_2\geq 1\\
2\mu f(n_1,n_2) & \text{if } n_1,n_2=0
\end{array}\right]
$$

It is sufficient to verify the following cases for nonnegativity.

Case a: $n_1, n_2 \geq 1$.

$$
\begin{aligned}
D_{12}T_3 f(n_1,n_2) &= \mu(D_1 f(n_1-1,n_2+1)-D_1 f(n_1-1,n_2)\\
&\quad +D_1 f(n_1,n_2)-D_1 f(n_1,n_2-1))\\
&= \mu(D_{12} f(n_1-1,n_2)+D_{12} f(n_1,n_2-1))\geq 0
\end{aligned}
$$

since $D_{12}f(n_1,n_2)\geq 0$ for $f\in V$.

Case b: $n_1\geq 1, n_2=0$.

$$
\begin{aligned}
D_{12}T_3 f(n_1,n_2) &= \mu(f(n_1,n_2+1)+f(n_1+1,n_2)\\
&\quad -f(n_1-1,n_2+1)-f(n_1,n_2))\\
&\quad -\mu_c(f(n_1,n_2)-f(n_1-1,n_2))\\
&\quad -(2\mu-\mu_c)(f(n_1+1,n_2)-f(n_1,n_2))\\
&= \mu(f(n_1,n_2+1)+f(n_1+1,n_2)\\
&\quad -f(n_1-1,n_2+1)-f(n_1,n_2)\\
&\quad +f(n_1,n_2)-f(n_1-1,n_2)\\
&\quad -f(n_1,n_2)+f(n_1-1,n_2))\\
&\quad -\mu_c(f(n_1,n_2)-f(n_1-1,n_2))
\end{aligned}
$$

$$
\begin{aligned}
& -(2\mu-\mu_c)(f(n_1+1,n_2)-f(n_1,n_2)) \\
= \;& \mu(f(n_1,n_2+1)+f(n_1+1,n_2) \\
& -f(n_1-1,n_2+1)-f(n_1,n_2) \\
& -f(n_1,n_2)+f(n_1-1,n_2)) \\
& -(\mu_c-\mu)(f(n_1,n_2)-f(n_1-1,n_2)) \\
& -(2\mu-\mu_c)(f(n_1+1,n_2)-f(n_1,n_2)) \\
\geq \;& \mu(f(n_1,n_2+1)+f(n_1+1,n_2) \\
& -f(n_1-1,n_2+1)-f(n_1,n_2) \\
& -f(n_1,n_2)+f(n_1-1,n_2)) \\
& -(\mu_c-\mu)(f(n_1+1,n_2)-f(n_1,n_2)) \\
& -(2\mu-\mu_c)(f(n_1+1,n_2)-f(n_1,n_2)) \\
= \;& \mu(f(n_1,n_2+1)+f(n_1+1,n_2) \\
& -f(n_1-1,n_2+1)-f(n_1,n_2) \\
& -f(n_1,n_2)+f(n_1-1,n_2) \\
& -f(n_1+1,n_2)+f(n_1,n_2)) \\
= \;& \mu(f(n_1,n_2+1)-f(n_1,n_2) \\
& -f(n_1-1,n_2+1)+f(n_1-1,n_2)) \\
= \;& \mu(D_1f(n_1-1,n_2+1)-D_1f(n_1-1,n_2)) \\
= \;& \mu(D_{12}f(n_1-1,n_2)) \geq 0.
\end{aligned}
$$

The first inequality follows from $D_1f(n_1,n_2)$ increasing in n_1 (property **P6**). The second inequality follows from $D_{12}f(n_1,n_2) \geq 0$ for $f \in V$.

Case c: $n_1 = 0, n_2 \geq 1$.

$$
D_{12}T_3f(n_1,n_2) \;=\; \mu(f(n_1,n_2+1)+f(n_1+1,n_2)
$$

$$
\begin{aligned}
& -f(n_1,n_2) - f(n_1+1,n_2-1)) \\
& -\mu_c(f(n_1,n_2) - f(n_1,n_2-1)) \\
& -(2\mu - \mu_c)(f(n_1,n_2+1) - f(n_1,n_2)) \\
= \;& \mu(f(n_1,n_2+1) + f(n_1+1,n_2) \\
& -f(n_1,n_2) - f(n_1+1,n_2-1) \\
& +f(n_1,n_2) - f(n_1,n_2-1) \\
& -f(n_1,n_2) + f(n_1,n_2-1)) \\
& -\mu_c(f(n_1,n_2) - f(n_1,n_2-1)) \\
& -(2\mu - \mu_c)(f(n_1,n_2+1) - f(n_1,n_2)) \\
= \;& \mu(f(n_1,n_2+1) + f(n_1+1,n_2) \\
& -f(n_1,n_2) - f(n_1+1,n_2-1) \\
& -f(n_1,n_2) + f(n_1,n_2-1)) \\
& -(\mu_c - \mu)(f(n_1,n_2) - f(n_1,n_2-1)) \\
& -(2\mu - \mu_c)(f(n_1,n_2+1) - f(n_1,n_2)) \\
\geq \;& \mu(f(n_1,n_2+1) + f(n_1+1,n_2) \\
& -f(n_1,n_2) - f(n_1+1,n_2-1) \\
& -f(n_1,n_2) + f(n_1,n_2-1)) \\
& -(\mu_c - \mu)(f(n_1,n_2+1) - f(n_1,n_2)) \\
& -(2\mu - \mu_c)(f(n_1,n_2+1) - f(n_1,n_2)) \\
= \;& \mu(f(n_1+1,n_2) - f(n_1,n_2) \\
& -f(n_1+1,n_2-1) + f(n_1,n_2-1)) \\
= \;& \mu(D_1 f(n_1,n_2) - D_1 f(n_1,n_2-1)) \\
= \;& \mu D_{12} f(n_1,n_2-1)) \geq 0.
\end{aligned}
$$

The first inequality follows from $D_2 f(n_1,n_2)$ increasing in n_2 (property **P6**).

The second inequality follows from $D_{12}f(n_1, n_2) \geq 0$ for $f \in V$.

Case d: $n_1 = 0, n_2 = 0$.

$$\begin{aligned}
D_{12}T_3f(n_1, n_2) &= \mu f(0,1) + \mu f(1,0) - \mu_c f(0,0) - (2\mu - \mu_c)f(0,1) \\
&\quad - \mu_c f(0,0) - (2\mu - \mu_c)f(1,0) \\
&\quad + \mu_c f(0,0) + (2\mu - \mu_c)f(0,0) \\
&= (\mu_c - \mu)(f(1,0) - f(0,0) + f(0,1) - f(0,0)) \\
&\geq 0
\end{aligned}$$

since $f(n_1, n_2)$ is increasing in n_1 and n_2 for $f \in V$ and the proof is complete.

$\square$

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Chapter 3

Auctions to Learn Consumer Demand for a Product with a Short Selling Horizon

3.1 Introduction

For a firm selling a new, innovative product, pricing is the crucial step to harvest the value created in developing and producing the new product. Yet pricing is particularly challenging for an innovative product, whose "newness" can naturally produce uncertainty about its exact demand distribution. Complicating matters further, obsolescence or competitive entry threats can shorten sales horizons during which the firm can capitalize on its product's market potential.

As a motivating example, consider SawStop, a small entrepreneurial startup, which recently developed a table saw with an innovative safety device that stops the saw blade the instant it contacts human skin. After showcasing its invention at tradeshows, initial orders were satisfied from SawStop at a price of approximately \$2,850 (Mehler 2005), whereas existing industry lines without the feature sold for between \$1000 and \$2,500. Eventually the firm began distributing to specialty woodworking stores, selling the saw at an MSRP of \$3,270 (Johnson 2006). While SawStop is the first saw with such safety features, an industry trade group, representing large tool manufacturers (Black & Decker, Bosch, Ryobi, and others), announced that it expects to have even better guarding mechanisms ready by sometime in 2007 (Skrzycki 2006). With such an innovative product, SawStop had to estimate consumers' willingness to pay for this breakthrough safety feature, but do so quickly given the looming threat of competitive entry.

In this paper we develop a framework and analysis of how a firm, with a product developed and ready to sell but unsure of consumer demand (willingness to pay), can strike a balance between two competing desires. On the one hand, the firm wishes to spend time gathering information on consumer demand with which to determine a fixed price, and on the other hand, the firm wishes to start mass market sales quickly to take advantage of the short sales horizon. Because the product has already been developed and is ready for sale, the firm is able to use a test market approach to gathering data. Unlike a traditional test market that requires inventory positioned in test stores and tightly controlled localized conditions (such as prices), we propose an approach that utilizes online auctions as the test market channel. Online consumer auctions are popular and growing – eBay had 222 million registered users in 2006 (eBay.com 2007), and by 2010 the online auction industry is expected to reach $65 billion in sales (Johnson and Tesch 2005). In addition to being relatively easy to set up and run, the key benefit of an online auction test market is the ability to observe consumer bids instead of just purchase/no purchase decisions.

In our model we divide the selling horizon into two phases: an initial phase in which the product is sold exclusively by auction and demand information is gathered; and a secondary stage in which a fixed price is set and the mass market is reached via posted price retailers (e.g., nation-wide rollout to retailers). To our knowledge, this study is the first to analytically model auctions to inform a mass-market, fixed-price selling phase. Balancing the firm's two competing desires, between information on the one hand and more time to exploit the market on the other, is the crux of this research problem, boiling down to a stopping time decision on when to abandon the auction phase and begin the mass market sales phase.

In making this stopping time decision, we find that the firm must pay close attention to several key factors. First is its point estimates of purchase probabilities at various prices, which are driven by observations of consumer bids in the auctions. Second is the anticipated shape of the mass-market sales trajectory (which reflects factors such as how the product will diffuse through the marketplace due to innovative and imitative purchases, Bass 1969). Third is the amount of time remaining in the selling horizon, which, depending on the shape of the sales trajectory, might encourage or discourage the use of auctions for market research. Furthermore, in using auctions for market research in this setting, the firm can use a proactive auction design that eases solution of the inverse problem between bids and willingness to pay, facilitating purchase probability estimations.

Below we flesh out more context for this work, beginning with background in auctions for market research and then moving to aspects of dynamic learning and stopping time literatures.

3.1.1 Auctions for Market Research

A great deal of experimental and empirical work has been conducted using auctions as a means for market research into willingness to pay, mostly in laboratory settings. A seminal laboratory study by Hoffman et al. (1993) using products in the packaged beef industry has been followed by other studies examining willingness to pay for new products, e.g., pesticide-free fruit (Roosen et al. 1998) and milk produced with the aid of bovine growth hormone (Fox et al. 1994). Experimental economists have widely tested how subjects behave in various forms of laboratory auctions (see the text by Kagel and Roth 1995), as well as how various auctions perform at eliciting "home-grown" (not induced by the experimenter) willingness-to-pay information (Ruström 1998, Noussair et al. 2004). Econometric analyses of empirical data have taken bid data from auctions in the field and reverse engineered it to reveal information on participants' willingness to pay; see, for example, the recent text by Paarsch and Hong

(2006).

Inspired by this lineage of experimental and empirical auction work to estimate willingness to pay, our paper develops an analytical (as opposed to experimental or empirical) model of how online auctions might be used for demand learning during the beginning of a short selling horizon. In an empirical study, Paarsch (1997) used historical auction field data to suggest an optimal reserve price for government timber auctions. While similar in spirit to our work, we treat the amount of data gathered as a decision variable controlled by the firm, who trades off more data (and a better informed fixed price) against more rapid entry to a mass market having a short selling horizon. Our use of a segmented explore then exploit approach also distinguishes our work from two other streams of analytical research on using auctions for demand learning. In operations management, Pinker et al. (2007) model how observations of previous auctions can be used to inform future auctions' lot sizes. In computer science, approximation algorithms have been developed to decide prices in so-called on-line auctions whereby bidders arrive sequentially and upon bidding the firm must immediately decide, based on past bid data, whether to reject or accept the bidder's offer (Bar-Yossef et al. 2002). At the other extreme, for the case in which the firm holds a single auction in which all potential consumers in the market participate, but the firm does not know the consumer willingness-to-pay distribution, Segal (2003) creates a mechanism whereby the price for each bidder is set based on a demand distribution inferred from other bidders' bids.

In our study, the firm uses auctions for the express purpose of eliciting willingness-to-pay information from consumers. Using selling mechanisms in such marketing research experiments has the advantage that actual purchasing decisions can be tested (as opposed to hypothetical surveys). Auctions, in particular, endogenously reveal the demand level at various price levels, as opposed to simulated store experiments, which only reveal demands at whatever fixed price the experimenter has set. But, in general, studies of auction experiments (in the lab or field) to elicit consumers' willingness to pay, of particular interest to marketers, have not paid attention to issues such as data censoring due to perceived outside options, affiliated beliefs about the value of such outside options, or affiliated beliefs about the quality of the tested product itself (Harrison et al. 2004). Such issues, if not controlled with the test market design, should be addressed when reverse engineering demand information from collected data. However, from a practical point of view it is likely easier to attempt to control for these factors via a proactive approach to test market design, and this motivates the second price sealed-bid auction with rebate introduced in §4.

The second price sealed-bid auction with rebate is a first step at mechanism design for the express purpose of demand curve learning in field auctions to inform a fixed price that will be available to consumers later. To our knowledge it is the first auction mechanism for market research designed to take into account a future fixed price purchasing opportunity.

That said, our results characterizing how the firm balances the tradeoffs between more data and a shorter sales horizon, i.e., the stopping time decision, are robust to whatever format of auction used, provided that the inverse problem mapping bids to willingness to pay is somehow solved. These results are related to explore and exploit stopping time literature that we discuss next.

3.1.2 Explore and Exploit

Once a product is ready for sale, demand information can be gathered by observing actual sales under different conditions, for example, by varying posted prices (see Balvers and Cosimano 1990 and references to the early literature therein, Aviv and Pazgal 2005, Lin 2006), inventory levels (e.g., Scarf 1959, Lovejoy 1990, Lariviere and Porteus 1999, Chen and Plambeck 2005), or product assortment (Caro and Gallien 2007). Our approach differs from these dynamic learning papers. First, we use auctions as a sales channel meant to gather demand information to set a fixed price. Consumer bids in auctions offer more detail on valuations than purchase/no purchase responses to a fixed price. Furthermore, auction observations are not censored by inventory stockouts.

Second, in the demand learning literature cited above, the firm controlling pricing (or inventory) faces a tradeoff between immediate gains by myopic control and future gains by control which aids learning. For instance, higher posted prices allow the firm to test the high end of the willingness-to-pay distribution at high per-unit profits, but censor data from the low end. In our model the firm's tradeoff is similar in spirit but quite different in structure from those above. In particular, our firm faces a tradeoff between short-term losses of product launch delay and long-term gains from demand learning, where after making the decision to stop auctioning (the control decision in our model), the firm sells via fixed price and learning ceases. Thus, rather than continual learning over the short horizon by adjusting prices (or inventory) upwards and downwards in response to demand history, we adopt a segmented approach more akin to test market research methods whereby a pure explore phase (auctions) is followed by a pure exploit phase (fixed price).

In introducing a stopping time problem framework for the use of auctions for market research, our work is related to sequential learning models in economics and management science. In the economics literature, for example, studies have analyzed how the timing of investments by individual firms facing uncertainty affect aggregate innovation diffusions (Jensen 1982) and economic investment cycles (Bernanke 1983). In the management science literature, McCardle (1985) studies the adoption of a new technology option whose profitability can be learned, with the firm sequentially deciding whether to adopt, reject with a fixed payoff, or continue learning at a cost. In our setting, the firm chooses among N "options" (prices) which have uncertain but correlated values. Lippman and McCardle (1991) study a firm facing an infinite number of options of uncertain but independent values; the firm learns about each option by sequentially collecting costly information, but each option is examined one at a time and either accepted or permanently discarded. In contrast, a main benefit of using auctions is that our firm can simultaneously gather information on all N prices by examining bids. Kornish and Keeney (2008) analyze simultaneous data collection on two, independent, options, modelling the choice of which of two influenza vaccines to produce before a flu season. Our model's short selling season context implies a finite horizon model, as does Kornish and Keeney's production planning setting. One of the key differences between our work and the literature is that the previous papers all assume that the parameters are independent. In our model, one observation will change all of the parameters in a way which is interrelated.

As is common in the sequential learning literature, these papers, and ours, specify threshold results of the following type: "adopt" an option if its expected value based on current information is high enough. We model the cost of information gathering as the implicit cost of delaying product launch, which in turn depends upon the evolution of demand from the moment of launch up to end of the selling horizon; accordingly, we examine how threshold results depend on demand evolution aspects, such as product diffusion (Bass 1969).

The remainder of the paper is organized as follows. The model and its assumptions are described in §3.2. The main results of the chapter, characterizing the firm's dynamic decision policy on when to continue the auction phase and when to commence the mass-market phase, are contained in §3.3. Subsection 3.3.3 discusses the special case when fixed-price sales follow a diffusion process per the Bass model. Section 3.4 concludes. Proofs of results are included in the Appendix.

3.2 The Model

We model a risk-neutral, revenue-maximizing firm selling a product over a finite horizon of length T. In this stylized model, T can be thought of as the demand window for the new product (Dolan 1993, Cohen et al. 1996), and could model the time to product obsolescence, the length of the selling season for a fashion product, or the time until low-cost imitator entry dissolves the firm's ability to charge a profitable margin for an innovative product. The firm is uncertain about consumers' willingness-to-pay, or valuation distribution, for the product. More precisely, for N possible prices ordered $p_1 < p_2 < \cdots < p_N$, the firm is uncertain of w_i, the true fraction of customers whose valuation is at least p_i. As is standard in sequential learning problems (McCardle 1985, Kornish and Keeney 2008), we assume that the true w_i's are static over the horizon, although the firm's beliefs about the w_i's may change based on its observations. Each individual consumer's valuation for the product is assumed to be private, independent, and identically distributed.

The selling horizon is divided into an auction phase and a fixed price phase. During the auction phase, the firm updates its beliefs about consumer demand based on data gathered from the auctions. The firm chooses the time at which to abandon the auction learning phase and switch to selling to the mass market via a fixed price. We begin by describing the fixed-price phase.

3.2.1 Fixed-Price Phase

Once a fixed price is set, say at p_i, at some time, $t \in [0, T]$, the mass market sales process begins. We envision this process as capturing sales in whatever fixed price retail channels the firm chooses to use (e.g., brick and mortar stores, catalogs, internet sales). We define m as the *total market potential*, that is, the size of the market segment for which the product is targeted. (For example, for SawStop's innovative safety saw, the target market could be a segment from the population of private, commercial, and educator woodworkers.) Of course, not every consumer in the targeted market would purchase the product; only those consumers whose valuation for the product equals or exceeds the price, p_i, would make a purchase. If w_i is the expected fraction of consumers willing to purchase the product at price p_i, then we say the *market potential* for p_i is $w_i m$.

When the firm sets the price at p_i, it cannot necessarily expect to sell $w_i m$ units before the end of the time horizon. The *rate of market saturation* that occurs z time units after market entry is denoted by $a(z)$, where $\int_{z=0}^{T-t} a(z)dz \leq 1$ represents the total fraction of the market that can be served over horizon $[t, T]$, for t the market entry time and T the end of the horizon. For example, $a(z) = \lambda \leq 1/T$ for $z \in [0, T]$ corresponds to constant market saturation rate of λ at the fixed price channel(s) – stores, web sites, etc. – yielding an expected sales rate of $w_i m \lambda$. While $a(z)$ could describe a constant expected rate of sales, $a(z)$ could also be a more general function of time; one such model, empirically shown to have descriptive and predictive power for the growth and decline of new product market saturation rates over time, is the Bass diffusion model (Bass 1969, Mahajan et al. 2000). For generality, we will leave the form of $a(z)$ unspecified. One exception is §3.3.3, an illustrative section which applies our results to the case of Bass diffusion. The total number of sales expected over the horizon $[t, T]$ is $w_i m \int_{z=0}^{T-t} a(z)dz$. However, a discount rate of r is applied to future revenue. For convenience we define the *market size*

$$M(t) \triangleq m \int_{z=0}^{T-t} a(z)e^{-rz}dz, \tag{3.2.1}$$

which can be thought of as the discounted revenue the firm would gain from time t until the end of the horizon if every arriving customer purchased the product for \$1. Given p_i and expected purchase fraction w_i, the expected discounted future revenue accruing to the firm upon stopping the auction phase and setting a fixed price at time

t is equal to $p_i w_i M(t)$.

3.2.2 Auction Phase

The auction phase starts at the beginning of the horizon, i.e., at time $t = 0$. During this phase, the firm only sells the product via fixed-duration auctions. We normalize T such that an auction lasts 1 period. The demand data gathered during an auction which begins at time $t-1$ and ends at time t is summarized by a signal, s_t, and the information captured by auctions up to time t is expressed by a sufficient statistic, $\mathcal{B}_t$, which is a function of $(s_1, s_2, \ldots, s_t)$ (see DeGroot 1970 for a discussion on sufficient statistics). The stochastic processes which generate such data are left general, with exceptions made explicit when needed (Corollaries 1 and 3, and Proposition 2). The expected fraction of customers who will purchase at prices less than or equal to p_i, conditional on sufficient statistic $\mathcal{B}_t$, is expressed by $w_i(\mathcal{B}_t) = E[w_i|\mathcal{B}_t]$. The firm's initial demand information at the beginning of the horizon is described by $\mathcal{B}_0$. For generality, we do not specify the updating scheme used by the firm, but do assume that point estimates of purchase probabilities are unbiased and take all information available at time t into account:

$$E[w_i(\mathcal{B}_{t+1})|\mathcal{B}_t] = w_i(\mathcal{B}_t) \quad \text{for all } i = 1, \ldots, N. \tag{3.2.2}$$

That is, point estimates are martingale. Note that all Bayesian updating schemes satisfy the martingale assumption, but this need not be true in general. For example, exponential smoothing, in which more recent data are given larger weights in determining beliefs, does not satisfy the martingale assumption. Section 3.3.1 provides an example using Bayesian updating and a multidimensional Beta-multinomial information structure.

We have not assumed a particular auction format or set of bidder beliefs and behaviors for the auction phase; all we assume is that, following each auction, the firm is able to update its purchase probabilities for each price based on the data signals it gathers during the auction. In §4 we propose a possible auction format, specify accompanying assumptions on bidder behavior, and discuss how the updating problem – essentially an inverse problem mapping bids to willingness to pay – could be solved. Other auction formats, appropriate to other assumptions on bidder behavior, could exist – for example, while §4 assumes rational behavior of bidders subject to that section's assumptions, other authors have examined computing willingness to pay from auction data without assuming much in the way of structural bidder rationality (e.g., Chan et al. 2007).

3.2.3 Firm's Stopping Time Problem

Since the time horizon is limited, and any amount of time spent holding auctions subtracts away from the time to sell to the mass market, it is necessary to stop holding auctions at some point. The firm must choose when to stop gathering demand information via auctions and commit to a fixed price for mass-market sales. Because each auction lasts one period, we can cast the firm's problem as a discrete time stopping problem with information updating. At stage t, the firm decides between $N+1$ alternatives: either commit to one of N fixed prices, $\{p_i\}_{i=1}^{N}$, which has an expected discounted payoff of $p_i w_i(\mathcal{B}_t)M(t)$ over the interval $[t,T]$, or continue auctioning for at least one more period to gather additional demand information. $\delta \triangleq e^{-1r}$ is the discount applied to a payoff delayed one period into the future. Accordingly, the firm solves the following dynamic program.

$$J_t(\mathcal{B}_t) = \max\Big\{\max_j\{p_j w_j(\mathcal{B}_t)M(t)\},\ \delta E[J_{t+1}(\mathcal{B}_{t+1})|\mathcal{B}_t]\Big\}, \tag{3.2.3}$$

with the boundary condition $J_T(\mathcal{B}_T) = 0 \; \forall \; \mathcal{B}_T$. The first expression in equation 3.2.3 is the value of stopping the auction phase, choosing the best price, and entering the mass market. The second term is the expected value of continuing the auction phase for one more time period.

Several modeling assumptions tacit in the above dynamic program warrant discussion before moving on. First, the variable cost to run an auction (running the auction website) is zero. We relax this assumption in §3.3.4 and examine the case with nonzero auctioning costs. Second, the model assumes the product's marginal cost is zero; a positive, constant marginal cost could easily be incorporated, with the expected fixed-price profit of price p_i becoming $(p_i - [\text{marginal cost}])w_i(\mathcal{B}_t)M(t)$.

Third, we do not include inventory holding costs, and it is assumed that the firm has sufficient capacity or inventory to meet its demand. This is for simplicity, in order to focus on demand learning for a new product, rather than focusing on issues such as managing inventory before and after a new product launch. Two papers, Kumar and Swaminathan (2003) and Ho et al. (2002), have focused on the latter, with a Bass model backdrop. Both introduce optimal (or near-optimal) policies whereby a capacitated firm delays launching the product to build an initial inventory stockpile that mitigates shortages during the selling horizon. The agreement between this type of policy on the one hand, and our framework of launch delay for learning on the other, suggests that omitting capacity and inventory decisions from our analysis would not greatly jeopardize our qualitative insights about demand learning; in fact, concerns such as capacity limitations may prompt the firm to delay mass-market entry for reasons beyond demand learning alone, allowing it to gather even more demand information than it otherwise would.

Fourth, we assume that the mere presence of a test market does not appreciably affect the underlying demand landscape for the product. That is, demand information can be gathered from the test-market auctions without influencing the characteristics of the demand, namely the size $M(t)$ of the potential mass market that can be reached (i.e., given the opportunity to purchase) during $[t, T]$, and the unknown, underlying distribution of such consumers' willingness to pay. Intuitively, any test-market sales (via auctions or any other means) may drive up demand by helping to spread positive word of mouth for a product, or have an opposite effect if the product is not well received. We leave such issues to future work, and focus here on learning a latent demand landscape that is unaffected by the fact that we examine the demand with a test market. We observe that while demand-influencing effects of test markets may tend to be encouraged by a large number of test-market sales, auctions tend in the opposite direction because they sell relatively few items. Note that a single-item auction may have many bidders but still sell only one item.

Finally, we do not include revenue from the test market within the dynamic program. Effectively, this assumes that the test market auction sales will be dwarfed by mass-market, posted-price sales. To put this into perspective, if a firm sold 40,000 units in a year via the mass market (e.g., through a nation-wide rollout to retailers), such mass-market sales would occur at an average rate of over 100 units per day. Intuitively, such a firm is unlikely to indulge in continuing the auction phase simply to capture additional auction revenue, which comes in at a much slower rate (auctions held by eBay and other retailers can last over a day, see Lucking-Reiley 2000). Despite revenue from the auctions not being a key concern, they do offer pricing information for the many units that will be sold during the mass-market phase. In summary, we model auctions as market research opportunities prior to mass-market sales of the product. In this spirit, the auction format for market research introduced in the next chapter seeks to maximize information gleaned from the auctions, rather than maximizing per-item auction revenue. (In contrast, the latter is traditionally the objective of a profit-maximizing auctioneer in the auctions and operations literature.)

3.3 Analysis of Stopping Time Problem

A firm armed with a method of inverting bids to willingness to pay is in a position to collect data and update its purchase probability estimates for the N prices. In general, the auction format or assumptions could require non-identity inversion mappings when recovering willingness to pay from observed auction data. However, for convenience we will refer to bids and willingness to pay interchangeably when discussing data gained during the auction phase (this interchange will only be precise in some settings, e.g., the second price with rebate auction format and assumptions described in §4).

3.3.1 Examples

We begin by providing simple, illustrative examples of updating and the stopping problem. We first illustrate how Bayesian updating converts bids to purchase probability estimates when the prior distribution of purchase probabilities comes from the multidimensional Beta family. Then, we provide a simple example in which the firm chooses to continue holding auctions rather than stopping and committing to a fixed price.

Multidimensional Beta-multinomial Bayesian updating. For a set of non-negative parameters $\gamma_0, \gamma_1, \ldots, \gamma_N$, the multidimensional Beta$(\gamma_0, \ldots, \gamma_N)$ distribution over non-negative random variables $x_0, \ldots, x_N$, $\sum_i x_i = 1$, is defined as

$$f(x_0, x_1, \ldots, x_N) = x_0^{\gamma_0 - 1} x_1^{\gamma_1 - 1} \cdots x_N^{\gamma_N - 1} \frac{\Gamma(\gamma_0 + \gamma_1 + \cdots + \gamma_N)}{\Gamma(\gamma_0)\Gamma(\gamma_1) \cdots \Gamma(\gamma_N)}$$

where

$$\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} dx.$$

For a positive integer argument k, $\Gamma(k) = (k-1)!$. The multidimensional Beta distribution can take on a vast variety of shapes, depending on the parameter values.

The multidimensional Beta distribution describes the joint probability of $N+1$ mutually exclusive events, where x_i is interpreted as the probability of the i^{th} event. For purchase probabilities $w_1 \geq w_2 \geq \cdots \geq w_N$, the value $w_i - w_{i+1}$ is the probability of the event that a random customer's valuation lies in interval $[p_i - p_{i+1})$. Suppose the initial joint prior distribution of the event probabilities for the $N+1$ intervals $[0, p_1), [p_1, p_2), \ldots, [p_N, \infty)$ follows Beta$(\gamma_{0,0}, \gamma_{1,0}, \ldots, \gamma_{N,0})$,

$$f(w_1, w_2, \ldots, w_N) =$$
$$(1-w_1)^{\gamma_{0,0}-1}(w_1-w_2)^{\gamma_{1,0}-1} \cdots (w_{N-1}-w_N)^{\gamma_{N-1,0}-1} w_N^{\gamma_{N,0}-1} \frac{\Gamma(\gamma_{0,0}+\gamma_{1,0}+\cdots+\gamma_{N,0})}{\Gamma(\gamma_{0,0})\Gamma(\gamma_{1,0})\cdots\Gamma(\gamma_{0,N})}.$$

The parameters of this distribution comprise the initial sufficient statistic, in other words, $\mathcal{B}_0 = (\gamma_{0,0}, \gamma_{1,0}, \ldots, \gamma_{N,0})$. The expected value of w_i based on this initial prior is $w_i(\mathcal{B}_0) = \sum_{j \geq i} \gamma_{j,0} / \sum_j \gamma_{j,0}$. We next discuss how the prior is updated based upon the number of bids observed in each of the $N+1$ price intervals. (Bayesian updating with a multidimensional Beta prior is also discussed in Silver 1965.)

Let b_t be the number of unique bids (observations) received during the t^{th} auction, lasting from $t-1$ to t. For $i = 1, \ldots, N-1$ let $b_{i,t}$ be the number of such bids that are in interval $[p_i, p_{i+1})$, with $b_{0,t}$ the number of bids in $[0, p_1)$ and $b_{N,t}$ the number in $[p_N, \infty)$. The true probability that a bid will lie in $[p_i, p_{i+1})$ is $w_i - w_{i+1}$. In other words, $(b_{0,t}, \ldots, b_{N,t})$ follows a multinomial distribution with $N+1$ different event possibilities and b_t trials.

After the t^{th} auction, the parameters of the prior distribution are updated. For each i, parameter $\gamma_{i,t} \triangleq \gamma_{i,t-1} + b_{i,t}$. That is, the number of observations made within the i^{th} price interval is added to the i^{th} interval's parameter. The new sufficient statistic is $\mathcal{B}_t = (\gamma_{0,t}, \gamma_{1,t}, \ldots, \gamma_{N,t})$. At stage t, the expected value of w_i, based on the current prior, is $w_i(\mathcal{B}_t) = \sum_{j \geq i} \gamma_{j,t} / \sum_j \gamma_{j,t}$. This estimate for the fraction of customers who will purchase at price p_i can essentially be thought of as the ratio of the number of "successes" (valuations observed in interval $[p_i, \infty)$), to the total number of observations, starting from an initial set of parameters $\mathcal{B}_0$.

Simple stopping problem. Assume that priors are updated according to the multidimensional Beta-multinomial information structure. Suppose $T = 2$, with $M(0) = 1000$, $M(1) = 970$, and $M(2) = 0$. The firm has an initial sufficient statistic $\mathcal{B}_0 = (0.5, 3, 1.5)$ with the following prices and priors: $p_1 = 14$, $p_2 = 32$, $w_1(\mathcal{B}_0) = 0.6$, $w_2(\mathcal{B}_0) = 0.3$. If the number of arrivals per auction is deterministic and set to 1, and $\delta = 1$, then at time $t = 0$, the expected value of continuing for one more auction is

$$
\begin{aligned}
&E\left[\max\left\{p_1 w_1(\mathcal{B}_1)M(1),\ p_2 w_2(\mathcal{B}_1)M(1)\right\}\middle|\mathcal{B}_0\right] \\
&\qquad = M(1)\Big((1 - w_1(\mathcal{B}_0))\max[7.00, 8.00] + (w_1(\mathcal{B}_0) - w_2(\mathcal{B}_0))\max[9.33, 8.00] \\
&\qquad\quad + w_2(\mathcal{B}_0)\max[9.33, 13.33]\Big) \;=\; 9,700.
\end{aligned}
$$

We have used the fact that $M(2) = 0$ and it is always optimal to stop if $t = 1$. On the other hand, if the firm stops at $t = 0$, the expected profit is

$$\max[p_1 w_1(\mathcal{B}_0)M(0),\ p_2 w_2(\mathcal{B}_0)M(0)] = 9,600.$$

In this case, at $t = 0$, it is optimal to continue the auction for one more period. A necessary, but not sufficient, condition for continuing the auction phase is that additional information can alter the optimal price choice. In this example, the valuation of the single additional observed bid is either below p_1, between p_1 and p_2, or above p_2, and the corresponding optimal price for each possibility is p_2, p_1, and p_2, respectively. In general, the incremental benefit of choosing a better price must be tempered with the loss in market size (e.g., $M(0)$ vs. $M(1)$).

3.3.2 Structure of the Stopping Problem

We now shed light on when is it optimal to stop the auction phase and turn to a fixed price. We begin by proving structural properties of the stopping time problem, and then characterize how the dynamics of mass-market sales affect the firm's decision.

Given the unrestricted richness of the sufficient statistic, $\mathcal{B}_t$, it would be nice to find a simple 1-dimensional statistic of $\mathcal{B}_t$ which admits a threshold optimal stopping policy. For example, the expected fraction of customers willing to purchase at price p_i, $w_i(\mathcal{B}_t)$, is a 1-dimensional statistic. However, given the sheer volume of potential histories, it is possible that vastly different sufficient statistics share the same purchase probabilities but have different stopping policies. For example, it could be that $w_i(\widehat{\mathcal{B}}_t) = w_i(\mathcal{B}_t)$, but $\widehat{\mathcal{B}}_t$ describes a history with 10 data points for which gathering more data is optimal, while $\mathcal{B}_t$ is describes a history with 10,000 data points for which new data is likely to have little impact on priors and stopping is optimal. Thus some common ground between histories, or their sufficient statistics, must be imposed before a 1-dimensional statistic can admit a stopping threshold. To this end, define a set of sufficient statistics at time t by Ω_t, which, for example, could be the set of sufficient statistics having 10 data points. If certain behavior over the sufficient statistics in set Ω_t exist, a threshold result can obtain. Below we establish conditions that ensure a stopping policy threshold exists over the $w_i(\mathcal{B}_t)$'s. In the proposition, $\mathcal{B}_{t+1}$ and $\widehat{\mathcal{B}}_{t+1}$ are the stage $t+1$ updates of stage t sufficient statistics $\mathcal{B}_t$ and $\widehat{\mathcal{B}}_t$, respectively. The term $\mathbf{w}_{-i}(\mathcal{B}_t) = (w_1(\mathcal{B}_t), \ldots, w_{i-1}(\mathcal{B}_t), w_{i+1}(\mathcal{B}_t), \ldots, w_N(\mathcal{B}_t))$ is the vector of point estimates excluding that of price i, and $\mathbf{p} = (p_1, p_2, \ldots, p_N)$ denotes the vector of prices.

Proposition 1. (Effect of point estimates.) *Let Ω_t be a set of sufficient statistics such that, for any $\mathcal{B}_t$, $\widehat{\mathcal{B}}_t \in \Omega_t$, if $w_i(\widehat{\mathcal{B}}_t) \geq w_i(\mathcal{B}_t)$ and $\mathbf{w}_{-i}(\widehat{\mathcal{B}}_t) = \mathbf{w}_{-i}(\mathcal{B}_t)$ then*

$$E[\phi(w_i(\widehat{\mathcal{B}}_{t+1}))|\widehat{\mathcal{B}}_t] \geq E[\phi(w_i(\mathcal{B}_{t+1}))|\mathcal{B}_t], \tag{3.3.4}$$

for any bounded, nondecreasing function ϕ. Then for any $\mathcal{B}_t \in \Omega_t$ there exist thresholds $h_{it}(\mathbf{w}_{-i}(\mathcal{B}_t), \Omega_t, \mathbf{p})$, $i = 1, \ldots, N$, such that if $w_i(\mathcal{B}_t) \geq h_{it}(\mathbf{w}_{-i}(\mathcal{B}_t), \Omega_t, \mathbf{p})$ it is optimal to stop the auction and commit to price p_i, and if $w_i(\mathcal{B}_t) < h_{it}(\mathbf{w}_{-i}(\mathcal{B}_t), \Omega_t, \mathbf{p})$ for all i, it is optimal to continue the auction phase. The thresholds, $h_{it}(\mathbf{w}_{-i}(\mathcal{B}_t), \Omega_t, \mathbf{p})$, are nondecreasing in p_j and $w_j(\mathcal{B}_t)$, $j \neq i$.

To interpret Proposition 1 it is helpful to consider the following example, stated as a corollary. Let b_{t+1} denote the number of bidders in the $t+1^{\text{st}}$ auction.

Corollary 1. *The results of Proposition 1 apply to the case where priors are updated using the multidimensional Beta-multinomial information structure and Ω_t is any set of sufficient statistics containing the same number of observations, if $[b_{t+1}|\mathcal{B}_t]$ and $[b_{t+1}|\widehat{\mathcal{B}}_t]$ have the same distribution for all $\mathcal{B}_t, \widehat{\mathcal{B}}_t \in \Omega_t$.*

By restricting to sufficient statistics having the same number of observations, the effect of the purchase probability point estimates on the stopping decision can be characterized. Corollary 1 states that, for the multidimensional Beta-multinomial information structure, if the firm finds it optimal to stop and choose price p_i at a certain history of S observations, then it is also optimal to stop and choose price p_i at any other S-observation history whose purchase probability point estimates are at least as large for price p_i but no larger for the other prices. To keep the comparisons between different histories apples-to-apples, the corollary requires that the number of bids received in the $t+1^{\text{st}}$ auction should not depend on the particular S-observation history at time t. (As just one example, this would hold for Poisson bidder arrivals even with an arrival rate that depends on t and S.) The upshot of Corollary 1 is that, while it is still possible that the firm chooses "incorrectly" (the optimal decision under limited information may not be that taken under full information), as it gathers more demand information the firm will eventually make a choice once a particular pricing option looks particularly dominant.

This threshold structure of Proposition 1 is illustrated in Figure 3.1 for a case in which the firm is deciding between just two prices. For readability, where the meaning is clear, we suppress the dependence of $w_i(\mathcal{B}_t)$ on $\mathcal{B}_t$. For values of (w_1, w_2) near the origin, the firm's optimal decision is to continue the auction phase. Holding w_2 fixed, when w_1 increases (moves to the right), the firm continues to prefer perpetuating the auction phase until w_1 becomes so large that it reaches the threshold $h_{1t}(w_2, \Omega_t, \mathbf{p})$, which is the boundary separating the 'continue the auction phase' region from the 'stop and choose p_1' region. At this point the firm prefers stopping and selecting price p_1, and the firm continues to prefer this for any further increase in w_1. This occurs even though continuing the auction phase and gathering more bid data would allow the firm to update its priors on both prices simultaneously. Finally, note that Proposition 1 does not guarantee uniqueness of an optimal price once the decision to stop has been made. The choice of price will be such that the value of $p_i w_i M(t)$ is maximized; it is conceivable that many choices of price may give the same maximal value, as represented by the diagonal line $w_1 p_1 = w_2 p_2$ dividing the stopping regions in Figure 3.1.

At its core, Proposition 1 examines the tradeoff between more information and faster entry to market. Condition (3.3.4) in the proposition is used to show that the more optimistic the firm is today about the market, the less the firm expects to regret entering. The underlying martingale assumption made on the updating process (equation (3.2.2)) also plays a role in Proposition 1, as it helps to ensure that it is better to stop today and set the price at p_i rather than wait until tomorrow and set the price at the *same* p_i. Proposition 1's result is similar in spirit to threshold results in other sequential learning studies (e.g., McCardle 1985, Kornish and Keeney 2008), although Proposition 1 treats an N-dimensional case, all N payoff options are updated in every period, and the options' payoffs are correlated rather than assumed to be independent.

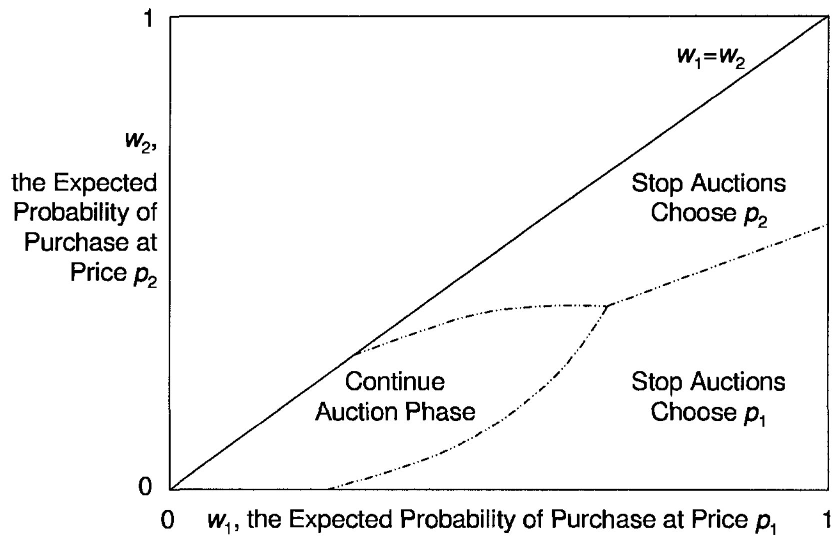

Figure 3.1: Structure of the optimal stopping time policy when there are two prices ($N = 2$).

At this point it is worth fleshing out the reason why thresholds in Proposition 1 depend on the set of histories, Ω_t. For this purpose, we use multidimensional Beta-multinomial updating as a backdrop. Consider a setting in which the firm is deliberating between two prices, as depicted in Figure 3.1. To gain intuition as to why the stopping decision depends on the set of sufficient statistics in addition to the purchase probability point estimates, consider two extreme cases, where Ω_t and Ω_t' are sets of sufficient statistics with 3 and 90 observations, respectively. Suppose $w_1(\mathcal{B}_t) = 2/3$ and $w_2(\mathcal{B}_t) = 1/3$. One might ask, for the stopping time decision, is it important whether or not $\mathcal{B}_t \in \Omega_t$ or Ω_t'? Consider what happens to the priors after one additional bid is received. If $\mathcal{B}_t \in \Omega_t$, the updated point estimates will be $(w_1(\mathcal{B}_{t+1}), w_2(\mathcal{B}_{t+1})) = (\frac{2}{4}, \frac{1}{4})$, or $(\frac{3}{4}, \frac{1}{4})$, or $(\frac{3}{4}, \frac{2}{4})$, each with probability 1/3, depending upon which interval, $[0, p_1)$, or $[p_1, p_2)$, or $[p_2, \infty)$, contains the new bid. On the other hand, if $\mathcal{B}_t \in \Omega_t'$, the updated priors will be $(w_1(\mathcal{B}_{t+1}), w_2(\mathcal{B}_{t+1})) = (\frac{60}{91}, \frac{30}{91})$, or $(\frac{61}{91}, \frac{30}{91})$, or $(\frac{31}{91}, \frac{61}{91})$, each with probability 1/3. Clearly, the number of preexisting observations (3 versus 90) has a dramatic effect on the expected change in priors caused by continuing the auction phase. In particular, it can be shown that the expected benefit of gathering exactly one additional auction's worth of data is always decreasing in the number of previous observations, for the multidimensional Beta-multinomial structure (see Proposition 7 in the Appendix).

Next, it is shown how the thresholds described in Proposition 1 are affected by the shape of the market size function, $M(t)$, as time elapses.

Proposition 2. *Suppose the stochastic processes generating auction data are stationary, i.e.,* $[\mathcal{B}_{t+k}|\mathcal{B}_t = \mathcal{B}]$ *has the same distribution as* $[\mathcal{B}_{s+k}|\mathcal{B}_s = \mathcal{B}]$ *for all* s, t, k*.*

1. **(Longer remaining horizon can encourage auctioning.)** *If the market size function,* $M(t)$*, is log-concave over* $[t-1, T-1]$*, then if the optimal decision at time* t *is to continue auctioning, the optimal decision under the same priors (sufficient statistic) at time* $t-1$ *is also to continue auctioning.*

2. **(Longer remaining horizon can discourage auctioning.)** *If the market size function,* $M(t)$*, is log-convex over* $[t-1, T-1]$*, then if the optimal decision at time* $t-1$ *is to continue auctioning, and with probability one the firm will enter before time* $T-1$*, the optimal decision under the same priors (sufficient statistic) at time* t *is also to continue auctioning.*

Proposition 2 shows that, in the firm's stopping time decision, the amount of time remaining in the horizon plays a key role. Part 1 says that, with a log-concave market size function, the more time remaining in the horizon, the more the firm is willing to delay the mass market entry in order to gather data. If a firm prefers to continue auctions with a given set of priors (or sufficient statistic) and remaining horizon, the firm would also prefer to continue auctioning with the same priors and more time remaining. As Figure 3.2 illustrates, the stopping regions become larger as the remaining time horizon shrinks. Part 1 requires log-concavity of the market size function $M(t)$. This assumption holds if $M(t)$ is concave, which in turn holds if

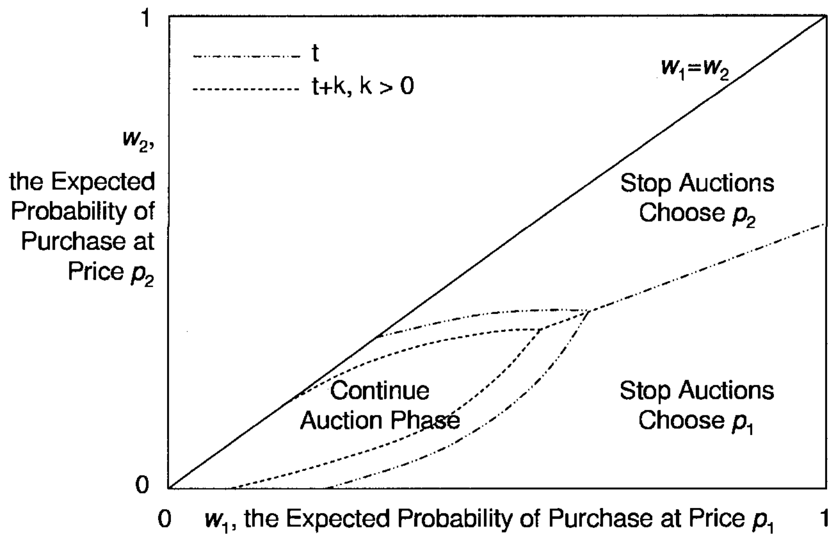

Figure 3.2: For a given set of sufficient statistics Ω, the continuation region shrinks as time elapses if $M(t)$ is log-concave. Below the diagonal, the $(- \cdot -)$ lines are $h_{it}(w_{-i}(\mathcal{B}), \Omega, \mathbf{p})$, and the $(\cdots)$ lines are $h_{i(t+k)}(w_{-i}(\mathcal{B}), \Omega, \mathbf{p})$, for $i = 1, 2$. For readability, dependence on sufficient statistic $\mathcal{B} \in \Omega$ is suppressed in the figure labels.

the rate of market saturation $a(z)$ is (weakly) decreasing in z, the time since mass market launch.

In contrast, if the rate of market saturation is highly convex in the time from mass market launch, early market entry is extremely valuable to the firm, in order to give the saturation curve time to pick up speed and make large gains late in the horizon. This is the intuition behind part 2 of Proposition 2, which says that with a log-convex market size function, the decision to continue auctioning with a long time horizon would, under the same priors, continue to be optimal for a shorter horizon. That is, under a highly convex market size function, the stopping regions can shrink even as the horizon becomes shorter — late in the horizon there is so little time to ramp up sales that the firm has less to lose by delaying entry, and accordingly is more willing to continue auctioning. That is, the market size will be roughly the same for a certain number of periods, so it is more important to set the price optimally than to go to market quickly.

Proposition 2 makes use of assumptions on the auction data process and the firm's entry decisions. First, the stochastic processes generating auction data are time insensitive, meaning that auctions are assumed to generate the same types and amounts of data, regardless of when they take place. This would not be the case, for example, if the firm knew bidding traffic would be very slow in early periods but increase dramatically for later periods. If auctioning offsets the cost of waiting only if traffic is high (i.e., auctioning can be beneficial only when the horizon is short but not when it is long), this would violate part 1 of the proposition if the market size is log-concave. Conversely, if traffic drops off in later periods, the stopping region might grow over time even for a log-convex market size function. Finally, part 2 requires that the firm wishing to continue auctioning at time $t-1$ enters the market by time $T-2$. While technical, this assumption is not unreasonable; if $M(t)$ is log-convex, the firm's market size decreases convexly as the firm delays its entry time towards T. To see why the entry assumption is important, suppose the firm continues at time $t-1$ due primarily to the profitable prospect of gathering exactly $T-t$ periods of data before entering. For a firm with the same priors but facing a horizon one period shorter, gathering $T-t$ periods of data would take it to the end of the horizon, at which point no sales can be made, lessening the appeal of continuing to auction at time t.

In summary, a longer time horizon can encourage or discourage furthering the auctioning phase, depending on the shape of the market size function. A log-concave market size function implies that as the remaining horizon shrinks, *more* sales would be lost by postponing entry to the market, and the impetus to enter the market becomes larger. In contrast, delay becomes *less* costly as the remaining horizon shrinks under a log-convex market size function, as the number of sales that delaying market entry sacrifices decreases as time elapses. The managerial insight is that, while a longer time horizon might naturally be seen to invite more data gathering under an optimal auction market research strategy, this intuition is sensitive to the trajectory of sales in the mass market.

Up to this point we have left the form of the market size function, $M(t)$, unspecified. However, this function could depend on any number of parameters. For example, the following subsection discusses the case in which mass market sales follow a Bass diffusion, where the relevant parameters capture the effects of two customer groups: innovators, who purchase based solely on external influences (e.g., mass market advertising), and imitators, who purchase based solely on internal factors (e.g., recommendations of past purchasers). Other parameters could model influences such as the number of stores willing to stock the product, or the geographic footprint of the firm's marketing campaign. The next result describes how the firm's stopping time decision can be sensitive to any such parameter, which we label as x.

Proposition 3. (Effect of market size parameters.) *Let x be some parameter of the market size function, $M(t)$, and let $I \subseteq \mathbb{R}$ be a subset of the real line. If $\frac{\partial}{\partial x}\left(\frac{M(s)}{M(t)}\right)$ is continuous and $\frac{\partial}{\partial x}\left(\frac{M(s)}{M(t)}\right)|_x \geq 0$ (≤ 0) $(= 0)$ for all $s \geq t$ and $x \in I$, the stopping region shrinks (grows) (does not change) as the market size parameter, x, increases within set I.*

Proposition 3 describes how the firm's market entry decision depends on parameters that influence the market size. The proposition implies that, if $M(s)/M(t)$ is monotonic in x for all $s \geq t$, the firm's stopping time decision is of a threshold type in the market parameter, x. For example, if $M(s)/M(t)$ is monotonically increasing in x, then holding t, $\mathcal{B}_t$, and all other parameter values fixed, once the firm prefers market entry with parameter value x_0 it continues to prefer market entry for any parameter value smaller than x_0.

The derivative condition describes how changing x affects the percent of market size lost by delaying entry. Intuitively, a parameter's impact on the stopping decision is affected by both today's market size and future market sizes. A main insight of the proposition is that ratios of the market sizes can be used to describe the effect on the stopping decision. To put the proposition in these terms, for any given parameter x that affects the market size, a larger value of x encourages market entry if increasing x always boosts the percentage of market size that would be lost by postponing entry. The opposite conclusion holds if larger x instead always reduces the percent of market size sacrificed by delaying entry, in which case increasing x favors further auctioning.

Proposition 3 may appear to require rather restrictive assumptions on the behavior of $M(t)$. However, the proposition's conditions can be established for natural market models and parameters. The results in the next section show precisely this for a market diffusion model, for which the parameters in question are coefficients of innovation and imitation. But first, a very simple application of Proposition 3 directly utilizes the fact that $M(t)$ is proportional to the total market potential, m.

Corollary 2. (Effect of total market potential.) *The firm's stopping decision does not depend on the particular size of the total market potential, m.*

In other words, Corollary 2 says that without loss of generality, the firm can ignore the particular size of m in its stopping time analysis since m is a scaler in the market size function, $M(t)$. This follows from Proposition 3, which says that the market size parameters affect the firm's stopping time behavior based only the ratio of current and future market sizes. While revenues depend strongly on the total possible market size m, the stopping decision depends on factors such as point estimates, time, and the shape of the sales trajectory (market size function shape) over time. Of course, this could change if fixed costs are associated with holding auctions; see §3.3.4 where we show that, under nonzero auctioning costs, the stopping regions would shrink as m grows.

3.3.3 Diffusion Sales Process

A widely applied model of new product adoption is the diffusion model first pioneered by Bass (1969) and since applied in many studies of new product introduction, forecasting, and sales management (Mahajan et al. 1990). The Bass model presents new product adoption as a diffusion process, akin to that of a contagious agent. In the model, there are two types of customers: innovators, who purchase based solely on external influences (e.g., mass market advertising), and imitators, who purchase based solely on internal factors (e.g., recommendations of past purchasers). In this subsection we will assume that the rate of market saturation can be described by the following differential equation, where A is the antiderivative of a:

$$a(s) = \frac{dA(s)}{ds} = \alpha(1 - A(s)) + \beta A(s)(1 - A(s)). \tag{3.3.5}$$

Here α is the coefficient of innovation and β is the coefficient of imitation as described by Bass. To solve (3.3.5), we set a boundary condition $A(0) = 0$. This accounts for the assumption (discussed on page 102) that the number of sales during the auction phase is negligible, and is consistent with imitation effects being insensitive to unsuccessful purchase attempts (Kumar and Swaminathan 2003). With this boundary condition, the diffusion equation becomes $A(s) = \left(1 - e^{-(\alpha+\beta)s}\right) / \left(1 + \beta e^{-(\alpha+\beta)s}/\alpha\right)$. Product adoptions, which follow trajectory $A(s)$, are taken to occur within the population of potential purchasers (e.g., Mahajan and Peterson 1978, Kalish 1985, Kalish and Lilien 1986). If the size of this population for fixed price p_i is $w_i m$, and mass-market sales begin at time t, then $w_i m A(T-t)$ is the expected number of sales for the horizon $[t, T]$. For the sake of tractability, we will ignore discounting, and set $M(t) = mA(T-t)$ for $t \leq T$. That is,

$$M(t) = \begin{cases} \frac{m\left(1-e^{-(\alpha+\beta)(T-t)}\right)}{1+\frac{\beta}{\alpha}e^{-(\alpha+\beta)(T-t)}} & \text{for } t \in [0, T] \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.6}$$

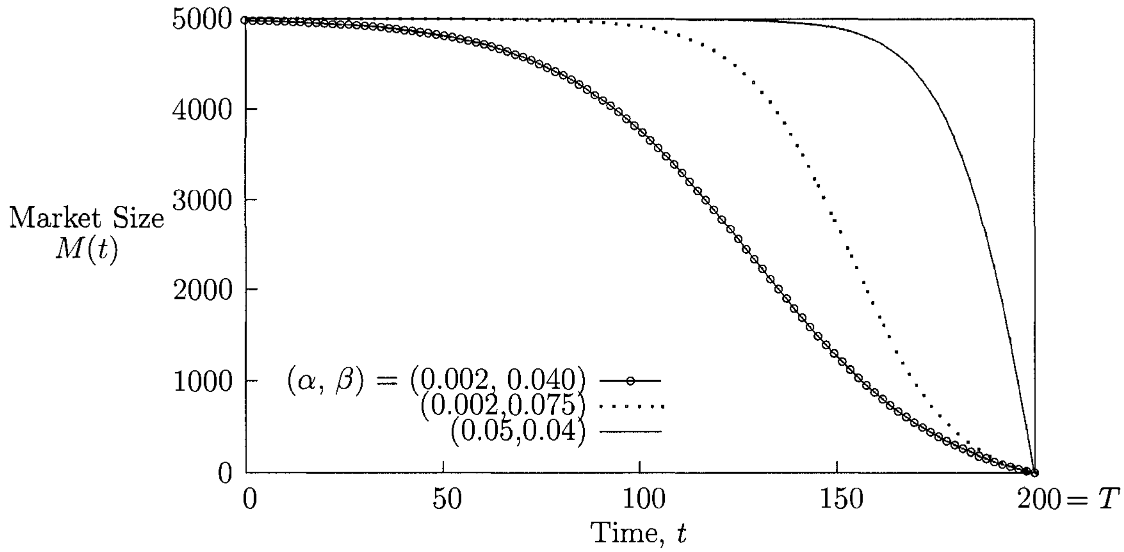

Figure 3.3: Sensitivity of sales trajectory to market parameters. A larger coefficient of imitation, β, enhances convexity in the sales trajectory due to "outbreak" effects. If the coefficient of innovation, α, outweighs the coefficient of imitation ($\alpha > \beta$), the sales trajectory is everywhere concave.

During a diffusion process, cumulative sales increase in the time from market launch. An example of the diffusion process is shown in Figure 3.3. How the shape of the sales trajectory depends on α and β can be understood by concentrating on each type of sales individually. Innovative sales increase at a decreasing rate (concavely) in the time since market launch, since their growth rate is proportional to the size of the untapped market (the first term on the righthand side of (3.3.5)). In contrast, the imitative sales rate is proportional to cumulative sales and the size of the untapped market, as captured by the second term on the righthand side of (3.3.5). When the size of the untapped market is still large, imitative sales grow at an increasing rate (convexly) as more cumulative sales spark more imitative purchases, analogous to the initial phase of an epidemiological outbreak. The convex imitative sales growth is only temporary, however, as a dwindling untapped market size inevitably drags the growth rates of both imitative and innovative sales to zero.

We next see how Propositions 1, 2, and 3 play out when sales follow a Bass diffusion. Since Proposition 1 does not depend on the market size function, its result clearly applies to the Bass diffusion setting. That is, when sales follow a Bass diffusion, stopping decision thresholds over purchase probability point estimates exist for updating structures satisfying the conditions of Proposition 1 (e.g., multidimensional Beta-multinomial updating, per Corollary 1). Our next result applies Proposition 2 to characterize how these thresholds change with time.

Corollary 3. (Longer remaining horizon encourages auctioning for Bass sales diffusion.) *Let the market size be given by* (3.3.6), *and suppose the stochastic processes generating auction data are stationary. If the optimal decision at time t is to continue auctioning, then the optimal decision at $t-1$ under the same sufficient statistic is also to continue auctioning.*

Corollary 3 says that a longer time horizon always favors further auctioning when sales follow a Bass diffusion and auction data processes are stationary. While the imitative effect can initially cause pronounced convexity in the sales curve (see Figure 3.3), no matter how large β is, the market size curve is still log-concave. That is, it never becomes "too convex," and a longer horizon always makes auctioning more desirable. Next, we further explore how the shape of the market size function impacts the firm's stopping decision. The following results apply Proposition 3 to describe how the stopping regions are affected by changes in the coefficients of innovation and imitation. First, we see that increasing the innovation coefficient always encourages delaying market entry.

Corollary 4. (Innovator effect encourages auctioning.) *Suppose the market size, $M(t)$, is given by* (3.3.6). *The stopping region always shrinks as the coefficient of innovation, α, increases.*

Corollary 4 further characterizes how the shape of the sales curve affects the firm's stopping time (market entry) decision. It says that for any coefficients of imitation and innovation, and for any time t and sufficient statistic, $\mathcal{B}_t$, the stopping region of Proposition 1 shrinks with the coefficient of imitation, α. Interestingly, increasing the coefficient of imitation can have the exact opposite effect, as the next result shows.

Corollary 5. (Imitator effect can discourage auctioning.) *Suppose the market size, $M(t)$, is given by (3.3.6), and let coefficient of imitation $\beta = \beta_0$. There exists a $t_0 \leq T$ such that for $t \geq t_0$, there exists a $\delta > 0$ such that the stopping region grows in β for $\beta \in (\beta_0 - \delta, \beta_0 + \delta)$. Furthermore, there exists α_0 such that $\alpha < \alpha_0$ implies $t_0 = 0$.*

The first part of Corollary 5 states that when we are sufficiently close to the end of the time horizon ($t \geq t_0$), the stopping region grows with the coefficient of imitation, as long as the coefficient of imitation is near the original value, β_0. The second part of the corollary states that the above property holds for all $t \in [0, T]$ when the coefficient of innovation is sufficiently small, relative to β_0. The key to Corollaries 4-5 lies in how strengthening innovation or imitation rates change the relative market loss caused by delaying entry, compared to immediate entry. For 'innovative' consumers, the innovative sales rate is proportional to only the size of the untapped market, which decreases as sales accumulate. Thus, the more sales can exhaust the potential market before reaching the end of the horizon (the higher the coefficient of innovation), the smaller the relative sales loss caused by delaying entry.

In contrast, the rate of imitative sales is proportional to cumulative sales as well as the size of the untapped market. The "contagion" effect initially causes the imitative sales rate to increase rapidly. During this phase, delaying entry reduces the snowball effect of sales. If sales do not have enough time to saturate the market before reaching the end of the selling horizon, under stronger contagion effects (higher coefficient of imitation), delaying market entry results in a larger loss relative to entering immediately. This is the case if the coefficient of innovation is very small (making the initial ramp-up very slow), or the time horizon is short ($t \geq t_0$). Like a sudden epidemic outbreak that wanes quickly due to a lack of susceptible individuals, with a long enough selling horizon or strong enough imitative or innovative effects, the market eventually becomes saturated before the end of the horizon. In this case, the more sales that accumulate before reaching the end of the horizon (the higher the coefficient of imitation), the smaller the relative sales loss caused by delaying entry. This explains why Corollary 5 only applies locally near β_0.

3.3.4 Nonzero Auctioning Costs

In this subsection we explore the case in which the variable cost to run auctions and gather bid data during a single period is c, paid at the end of each auction period. The dynamic program formulation (3.2.3) changes in the natural way, adding $-\delta c$ to the last term in the maximization (equation (3.4.20) in the Appendix). We have the following results.

Proposition 4. (Effects of point estimates and remaining time under nonzero auctioning costs.) *For the firm's stopping time decision under nonzero auctioning costs, Propositions 1 (thresholds in purchase probability point estimates) and part 1 of Proposition 2 (stopping regions grow with remaining time for log-concave market size function) hold as before.*

However, Proposition 3 (the effect of market size parameters) must be slightly changed to accommodate the effect of nonzero auctioning costs.

Proposition 5. (Effect of market size parameters under nonzero auctioning costs.) *Let x be some parameter of the market size function $M(t)$, and $I \subseteq \mathbb{R}$. If $\frac{\partial}{\partial x}\left(\frac{M(s)}{M(t)}\right)$ and $\frac{\partial}{\partial x}M(t)$ are continuous, and $\frac{\partial}{\partial x}\left(\frac{M(s)}{M(t)}\right)|_x$ and $\frac{\partial}{\partial x}M(t)\big|_x$ are both ≥ 0 (≤ 0) $(= 0)$ for all $s \geq t$ and $x \in I$, then the stopping region shrinks (grows) (does not change) as the market size parameter, x, increases within set I.*

Proposition 5 immediately implies that stopping regions shrink with the total potential market size m, which scales $M(t)$. The managerial insight here is that, when auctions are costly to operate, both shape *and* magnitude of the market size function are important. Intuitively, the magnitude is important to help offset the variable cost of running auctions, and shape is important to ensure that not too much of the market is sacrificed by delay. The importance of magnitude is what prevents part 2 of Proposition 2 from holding as before; we can have cases where, even with a log-convex market size function, the stopping region does not shrink with time. If the market size dips far below the auctioning cost, say $p_N M(t+1) < c$, then no matter what the shape of $M(t)$, continuing the auction phase at $t-1$ will not imply that under the same sufficient statistic continuing at t would be optimal.

Finally, as auctions become more costly, the firm naturally becomes more inclined to enter the mass market sooner:

Proposition 6. (Costlier auctioning encourages mass-market entry.) *The stopping region grows as c, the variable cost to run auctions and gather bid data during a single period, increases.*

3.4 Conclusions

This paper presents a framework and analysis for deploying online auctions as a demand-learning tool. By using online auctions for initial sales of a new product for which demand is uncertain, information on consumer willingness to pay can be gathered before a fixed price is set for the mass market (e.g., posted-price retailers). We study how long a firm should delay market entry in order to learn demand information: On one hand, delaying mass-market entry allows demand learning and enables a more effective pricing decision. But on the other hand, a finite sales horizon means that launch delays subtract from the time available to make sales in the mass market. We focus on closed-form characterizations of insights, in order to build intuition on how online auctions might best be used in this novel context.

The firm's decision about when to enter the mass market is modeled as a dynamic program, one trading off better demand information versus faster market entry. The two key elements of the model – the demand information accumulation process (i.e., auction format and associated information updating structure), and the mass-market sales process (i.e., shape of the mass-market sales trajectory), are left general, provided the information updating process is martingale. General insights are derived, and applied to infer implications for specific instances (such as Bayesian information updating, or diffusion-based market saturation processes).

The demand information gathering process informs the firm's predictions about the consumer's probability of purchase at N different price points. There is an appealing structure for the optimal policy: once the expected return of one price p (essentially, price times probability of purchase) emerges as sufficiently attractive relative to all other prices, the firm sets p as the fixed price and enters the mass market. This is captured by Proposition 1. The upshot is that, while it is still possible that the firm chooses "incorrectly" (the optimal decision under limited information may not be the same as under full information), as more demand information is gathered the firm will eventually make a choice once there is a pricing option that looks particularly dominant.

While the firm waits to discover which price becomes particularly attractive, it must also weigh the implications of the sales trajectory pattern for the mass market. When sales are anticipated to "snowball" dramatically in the time from mass-market launch, the firm may wish to forego prolonged demand information gathering and simply jump into the mass market quickly in order to allow sales sufficient time to ramp up prior to the end of the horizon. However, when sales growth is expected to need less ramping-up, or even decrease over time, the firm has less to lose by delaying its mass-market launch and auctioning for demand learning is more attractive. In fact, this latter case prevails for the classic and well-tested Bass model of new product diffusion, suggesting that for practical applications auctioning at the outset of a short selling horizon may indeed be attractive. These results are captured by Proposition 2 and Corollary 3. Additional results, Proposition 3 and Corollaries 4-5, characterize how entry decisions depend on other factors of the market trajectory. For example, when applied to the Bass model these results suggest that innovator effects encourage auctioning, while imitator effects can discourage it. In practice, because innovator effects may be possible to strengthen through activities such as increased advertising, a larger promotional budget could be seen as one way to "buy time" for the firm who wishes to delay mass-market launch to allow for further demand learning via auctions.

Mass-market fixed pricing helps focus our results on the main tradeoff between learning demand through auctions on one hand, and delaying market entry on the other (this tradeoff can become rather complex to characterize, see, for example, the proof of Corollary 1). Future work could perhaps extend the results to include additional complexities such as enhanced revenue capture through flexible posted pricing during mass-market selling. Flexible posted pricing would also extend the firm's learning opportunities into the mass-market phase. While in this paper we have focused on auctions as the demand-learning forum, examining the use of auctions in tandem with one or more other demand learning strategies, such as strategically adjusting posted prices to observe demand impacts during the mass-market phase, is an interesting, but daunting, opportunity for future work.

Appendix

Proof of Proposition 1

To prove existence of the thresholds, it is sufficient to show that, for $\mathcal{B}_t \in \Omega_t$, $p_i w_i(\mathcal{B}_t)M(t) - \delta E[J_{t+1}(\mathcal{B}_{t+1})|\mathcal{B}_t]$ is nondecreasing in $w_i(\mathcal{B}_t)$ for all t and i (prices). (Increasing $w_i(\mathcal{B}_t)$ can be thought of as shifting from $\mathcal{B}_t$ to $\widehat{\mathcal{B}}_t \in \Omega_t$, where $w_i(\widehat{\mathcal{B}}_t) > w_i(\mathcal{B}_t)$ and $\mathbf{w}_{-i}(\widehat{\mathcal{B}}_t) = \mathbf{w}_{-i}(\mathcal{B}_t)$.) First, it is shown that $p_i w_i(\mathcal{B}_t)M(t) - J_t(\mathcal{B}_t)$ is nondecreasing in $w_i(\mathcal{B}_t)$ by induction on t. Clearly this holds for $t = T$, since $J_T(\mathcal{B}_T) = 0$ and the first term is increasing in $w_i(\mathcal{B}_T)$. Assume that it holds for $t+1$ for some $t \leq T-1$. Now,

$$
\begin{aligned}
&p_i w_i(\mathcal{B}_t)M(t) - J_t(\mathcal{B}_t) \\
&\quad = p_i w_i(\mathcal{B}_t)M(t) - \max\Big[p_1 w_1(\mathcal{B}_t)M(t), \ldots, p_N w_N(\mathcal{B}_t)M(t), \delta E[J_{t+1}(\mathcal{B}_{t+1})|\mathcal{B}_t]\Big].
\end{aligned}
$$

On the righthand side, the result is trivially nondecreasing in $w_i(\mathcal{B}_t)$ if the maximum of the second term is one of the first N expressions. If the maximum of the second term is the last expression, then

$$
\begin{aligned}
&p_i w_i(\mathcal{B}_t)M(t) - J_t(\mathcal{B}_t) \\
&= p_i w_i(\mathcal{B}_t)M(t) - \delta E[J_{t+1}(\mathcal{B}_{t+1})|\mathcal{B}_t], \\
&= p_i w_i(\mathcal{B}_t)M(t) - \delta E[p_i w_i(\mathcal{B}_{t+1})M(t+1)|\mathcal{B}_t] \\
&\quad + \delta E[p_i w_i(\mathcal{B}_{t+1})M(t+1)|\mathcal{B}_t] - \delta E[J_{t+1}(\mathcal{B}_{t+1})|\mathcal{B}_t], \\
&= p_i w_i(\mathcal{B}_t)\big(M(t) - \delta M(t+1)\big) \qquad (3.4.7) \\
&\quad + \delta E[p_i w_i(\mathcal{B}_{t+1})M(t+1) - J_{t+1}(\mathcal{B}_{t+1})|\mathcal{B}_t].
\end{aligned}
$$

The first term of (3.4.7) is derived by noting that $E[w_i(\mathcal{B}_{t+1})|\mathcal{B}_t] = w_i(\mathcal{B}_t)$ by the martingale assumption, equation (3.2.2). This term is nondecreasing in $w_i(\mathcal{B}_t)$ since

$M(t)$ is nonincreasing in t. The expression $p_i w_i(\mathcal{B}_{t+1})M(t+1) - J_{t+1}(\mathcal{B}_{t+1})$ is nondecreasing in $w_i(\mathcal{B}_{t+1})$ by the induction hypothesis, and also clearly is a bounded function. By equation (3.3.4), the expectation of this expression, conditional on $\mathcal{B}_t$, is nondecreasing in $w_i(\mathcal{B}_t)$, and thus it has been shown that $p_i w_i(\mathcal{B}_t)M(t) - J_t(\mathcal{B}_t)$ is nondecreasing in $w_i(\mathcal{B}_t)$. The existence of the threshold function follows immediately from the fact that $p_i w_i(\mathcal{B}_t)M(t) - \delta E[J_{t+1}(\mathcal{B}_{t+1})|\mathcal{B}_t]$ is nondecreasing in $w_i(\mathcal{B}_t)$, which was just shown. Next, it is shown that the threshold function, $h_{it}(w_{-i}(\mathcal{B}_t), \Omega_t, \mathbf{p})$, is nondecreasing in $w_{-i}(\mathcal{B}_t)$.

The threshold function for price i must be nondecreasing in $w_j(\mathcal{B}_t)$, $i \neq j$. To see this, the threshold occurs when $p_i w_i(\mathcal{B}_t)M(t)$ is greater than or equal to all $p_j w_j(\mathcal{B}_t)M(t)$, $i \neq j$, and when $p_i w_i(\mathcal{B}_t)M(t)$ is greater than $\delta E[J_{t+1}(\mathcal{B}_{t+1})|\mathcal{B}_t]$. In the first case, the threshold must be nondecreasing in $w_j(\mathcal{B}_t)$. In the second case, it is necessary that the expectation, $E[J_{t+1}(\mathcal{B}_{t+1})|\mathcal{B}_t]$, is nondecreasing in $w_j(\mathcal{B}_t)$ for all j, which follows from a simple induction argument and equation (3.3.4). The behavior of the threshold in p_j, $j \neq i$, is similar. ∎

Proof of Corollary 1

Multidimensional Beta-multinomial updating satisfies the martingale assumption (3.2.2) by the definition of Bayesian updating and conditional expectation. On page 103 we discussed a sufficient statistic for multidimensional Beta-multinomial updating. Below we will use a slightly different sufficient statistic, using terminology from page 103. To this end, let $S_{j,t} \triangleq \sum_{i \geq j} \gamma_{i,t}$. Let $\mathbf{w} \triangleq (w_1, w_2, \ldots, w_N)$, and let $\mathbf{w}_{-i}$ be the vector excluding w_i.

Note that $\mathbf{S}_t \triangleq (S_{0,t}, S_{1,t}, \ldots, S_{N,t})$ is a sufficient statistic, and $w_i(\mathcal{B}_t = \mathbf{S}_t) = S_{i,t}/S_{0,t}$. Thus for two histories, $\mathcal{B}_t = \mathbf{S}_t$ and $\widehat{\mathcal{B}}_t = \hat{\mathbf{S}}_t$, having the same number of observations $S_{0,t}$, with $w_i(\widehat{\mathcal{B}}_t) \geq w_i(\mathcal{B}_t)$ and $\mathbf{w}_{-i}(\widehat{\mathcal{B}}_t) = \mathbf{w}_{-i}(\mathcal{B}_t)$, it must be that $\hat{S}_{i,t} \geq S_{i,t}$ and $\hat{S}_{j,t} = S_{j,t}$ $j \neq i$. In words, compared to $\mathbf{S}_t$, $\hat{\mathbf{S}}_t$ has some observations (bids) shifted from interval $[p_{i-1}, p_i)$ to $[p_i, p_{i+1})$.

Note that the joint prior distribution of $w_0 \geq w_1 \geq \cdots \geq w_{N+1}$ is multidimensional beta,

$$f(\mathbf{w}|\mathcal{B}_t) = \Gamma(S_{0,t}) \prod_{j=1}^{N} \frac{(w_j - w_{j+1})^{S_{j,t}-S_{j+1,t}-1}}{\Gamma(S_{j,t} - S_{j+1,t})}, \qquad \text{where } S_{N+1,t} \triangleq 0. \tag{3.4.8}$$

Integrating (3.4.8) over $w_i \in [w_{i+1}, w_{i-1}]$ shows that the marginal distribution, $f(\mathbf{w}_{-i}|\mathcal{B}_t)$, depends only on $\mathbf{S}_{-i,t}$, which from the arguments above equals $\hat{\mathbf{S}}_{-i,t}$ (the subscript $-i$ refers to all but the i^{th} element). Thus, we conclude that $f(\mathbf{w}_{-i}|\mathcal{B}_t) = f(\mathbf{w}_{-i}|\widehat{\mathcal{B}}_t)$. However, since

$$f(w_i|\mathcal{B}_t) = w_i^{S_{i,t}-1}(1 - w_i)^{S_{0,t}-S_{i,t}-1} \frac{\Gamma(S_{0,t})}{\Gamma(S_{i,t})\Gamma(S_{0,t} - S_{i,t})}, \tag{3.4.9}$$

it is easy to see that $\hat{S}_{i,t} > S_{i,t}$ implies $f(w_i|\mathcal{B}_t) \neq f(w_i|\widehat{\mathcal{B}}_t)$. To show that equation (3.3.4) holds (our desired result), we will use the notion of likelihood ratio ordering, denoted "$\leq_{LR}$" (see p12 of Müller and Stoyan 2002). By definition, for two random variables X and Y, $X \leq_{LR} Y$ if and only if $f_X(v)f_Y(u) \leq f_X(u)f_Y(v)$ for all $u \leq v$. It is easy to check using (3.4.9) that $[w_i|\mathcal{B}_t] \leq_{LR} [w_i|\widehat{\mathcal{B}}_t]$.

Towards showing (3.3.4) holds, we first characterize properties of $E[\phi(w_i(\mathcal{B}_{t+1}))|\mathcal{B}_t]$. Define $p_0 \triangleq 0$, $p_{N+1} \triangleq \infty$, $w_0 = 1$ and $w_{N+1} = 0$. Let K_j, $j = 0, \ldots, N$ be the number of bids received in the $t + 1^{\text{st}}$ auction that are greater than or equal to price p_j. Note that K_0 is the total number of bids received in the auction. (Thus K_0 is the same as b_t in the statement of Corollary 1.) Let $\mathcal{B}_{t+1} \triangleq (S_{0,t} + K_0, S_{1,t} + K_1, \ldots, S_{N,t} + K_N) \triangleq \mathbf{S}_t + \mathbf{K}$ denote the updated sufficient statistic after the $t + 1^{\text{st}}$ auction. We now show that $E[\phi(w_i(\mathcal{B}_{t+1}))|\mathcal{B}_t, K_0, \mathbf{w}]$ is nondecreasing in w_i for all K_0, $\mathbf{w}$, and $\mathcal{B}_t$. Let $0 \leq k_j \leq K_0$ be the number of bids in interval $[p_j, p_{j+1})$. Then

$$E[\phi(w_i(\mathcal{B}_{t+1}))|\mathcal{B}_t, K_0, \mathbf{w}]$$

$$= \sum_{\substack{k_0,\ldots,k_N \\ s.t. \sum_{j=0}^N k_j = K_0}} K_0! \prod_{j=0}^{N} \frac{(w_j - w_{j+1})^{k_j}}{k_j!} \phi(w_i(\mathbf{S}_t + \mathbf{K})) \tag{3.4.10}$$

$$= \sum_{R=0}^{K_0} \sum_{\substack{k_j,\ j \neq i-1,i \\ s.t. \sum_{j\neq i-1,i} k_j = K_0 - R}} K_0! \prod_{\substack{j=1,\ldots,N \\ j\neq i-1,i}} \frac{(w_j - w_{j+1})^{k_j}}{k_j!} \tag{3.4.11}$$

$$\times \sum_{k_{i-1}=0}^{R} \frac{(w_{i-1} - w_i)^{k_{i-1}}}{k_{i-1}!} \frac{(w_i - w_{i+1})^{R-k_{i-1}}}{(R-k_{i-1})!} \phi(w_i(\mathbf{S}_t + \mathbf{K})),$$

where for the second equality we have rewritten the terms using $R = k_{i-1} + k_i$. We differentiate the above with respect to w_i using the product rule.

$$\frac{dE[\phi(w_i(\mathcal{B}_{t+1}))|\mathcal{B}_t, K_0, \mathbf{w}]}{dw_i} = \sum_{R=0}^{K_0} \sum_{\substack{k_j,\ j \neq i-1,i \\ s.t. \sum_{j\neq i-1,i} k_j = K_0 - R}} K_0! \prod_{\substack{j=1,\ldots,N \\ j\neq i-1,i}} \frac{(w_j - w_{j+1})^{k_j}}{k_j!}$$

$$\sum_{k_{i-1}=0}^{R-1} \frac{(w_{i-1} - w_i)^{k_{i-1}}}{k_{i-1}!} \frac{(w_i - w_{i+1})^{R-k_{i-1}-1}}{(R-k_{i-1}-1)!}$$

$$\times \Bigg(\phi(w_i(\mathbf{S}_t + (K_0, \ldots, K_{i-1},\ K_{i-1} - k_{i-1},\ K_{i+1}, \ldots, K_N)))$$

$$-\phi(w_i(\mathbf{S}_t + (K_0, \ldots, K_{i-1},\ K_{i-1} - k_{i-1} - 1,\ K_{i+1}, \ldots, K_N))) \Bigg),$$

which is nonnegative since ϕ is nondecreasing and $w_i(\mathbf{S}_{t+1}) = S_{i,t+1}/S_{0,t+1}$. Finally, using the Corollary's assumption, $\Pr(K_0 = j|\mathcal{B}_t) = \Pr(K_0 = j|\widehat{\mathcal{B}}_t)$, and for convenience writing this probability as simply $\Pr(K_0 = j)$, we have

$$E[\phi(w_i(\mathcal{B}_{t+1}))|\mathcal{B}_t]$$

$$= \int_{\mathbf{w}} f(\mathbf{w}|\mathcal{B}_t) \sum_{j=0}^{\infty} E[\phi(w_i(\mathcal{B}_{t+1}))|\mathcal{B}_t, K_0 = j, \mathbf{w}] \Pr(K_0 = j) d\mathbf{w}$$

$$\leq \int_{\mathbf{w}} f(\mathbf{w}|\mathcal{B}_t) \sum_{j=0}^{\infty} E[\phi(w_i(\widehat{\mathcal{B}}_{t+1}))|\widehat{\mathcal{B}}_t, K_0 = j, \mathbf{w}] \Pr(K_0 = j) d\mathbf{w} \tag{3.4.12}$$

$$= \int_{\mathbf{w}_{-i}} f(\mathbf{w}_{-i}|\mathcal{B}_t) \int_{w_i|\mathbf{w}_{-i}} \Big(f(w_i|\mathcal{B}_t, \mathbf{w}_{-i})$$

$$
\begin{aligned}
& \sum_{j=0}^{\infty} E[\phi(w_i(\widehat{\mathcal{B}}_{t+1}))|\widehat{\mathcal{B}}_t, K_0 = j, \mathbf{w}] \Pr(K_0 = j)\Big) d\mathbf{w} \\
= & \int_{\mathbf{w}_{-i}} f(\mathbf{w}_{-i}|\mathcal{B}_t) \sum_{j=0}^{\infty} \int_{w_i|\mathbf{w}_{-i}} \Big(f(w_i|\mathcal{B}_t, \mathbf{w}_{-i}) \\
& E[\phi(w_i(\widehat{\mathcal{B}}_{t+1}))|\widehat{\mathcal{B}}_t, K_0 = j, \mathbf{w}] \Pr(K_0 = j)\Big) d\mathbf{w} \qquad (3.4.13) \\
= & \int_{\mathbf{w}_{-i}} f(\mathbf{w}_{-i}|\mathcal{B}_t) \sum_{j=0}^{\infty} E_{w_i|\mathcal{B}_t, \mathbf{w}_{-i}} \Big[E[\phi(w_i(\widehat{\mathcal{B}}_{t+1}))|\widehat{\mathcal{B}}_t, K_0 = j, \mathbf{w}] \\
& \times \Pr(K_0 = j)\Big] d\mathbf{w} \\
= & \int_{\mathbf{w}_{-i}} f(\mathbf{w}_{-i}|\widehat{\mathcal{B}}_t) \sum_{j=0}^{\infty} E_{w_i|\mathcal{B}_t, \mathbf{w}_{-i}} \Big[E[\phi(w_i(\widehat{\mathcal{B}}_{t+1}))|\widehat{\mathcal{B}}_t, K_0 = j, \mathbf{w}] \\
& \times \Pr(K_0 = j)\Big] d\mathbf{w} \qquad (3.4.14) \\
\leq & \int_{\mathbf{w}_{-i}} f(\mathbf{w}_{-i}|\widehat{\mathcal{B}}_t) \sum_{j=0}^{\infty} E_{w_i|\widehat{\mathcal{B}}_t, \mathbf{w}_{-i}} \Big[E[\phi(w_i(\widehat{\mathcal{B}}_{t+1}))|\widehat{\mathcal{B}}_t, K_0 = j, \mathbf{w}] \\
& \times \Pr(K_0 = j)\Big] d\mathbf{w} \qquad (3.4.15) \\
= & \int_{\mathbf{w}} f(\mathbf{w}|\widehat{\mathcal{B}}_t) \sum_{j=0}^{\infty} E[\phi(w_i(\widehat{\mathcal{B}}_{t+1}))|\widehat{\mathcal{B}}_t, K_0 = j, \mathbf{w}] \Pr(K_0 = j) d\mathbf{w} \\
= & E[\phi(w_i(\widehat{\mathcal{B}}_{t+1}))|\widehat{\mathcal{B}}_t].
\end{aligned}
$$

The inequality in (3.4.12) follows from substituting $\hat{\mathbf{S}}_t$ for $\mathbf{S}_t$ in equation (3.4.10) and noting that $\phi(w_i(\mathbf{S}_{t+1}))$ is nondecreasing in $S_{i,t+1}$. The interchange of the integral and the infinite summation in (3.4.13) is allowed by the dominated convergence theorem since

$$
\left| \lim_{n\to\infty} \sum_{j=0}^{n} f(w_i|\mathcal{B}_t, \mathbf{w}_{-i}) E[\phi(w_i(\widehat{\mathcal{B}}_{t+1}))|\widehat{\mathcal{B}}_t, K_0 = j, \mathbf{w}] \Pr(K_0 = j) \right| < f(w_i|\mathcal{B}_t, \mathbf{w}_{-i}) Q
$$

for some $Q < \infty$ and $f(w_i|\mathcal{B}_t, \mathbf{w}_{-i})Q$ is measurable. The equality in (3.4.14) follows by $f(\mathbf{w}_{-i}|\mathcal{B}_t) = f(\mathbf{w}_{-i}|\widehat{\mathcal{B}}_t)$, which was established above. To understand the inequality in (3.4.15), first note that $[w_i|\mathcal{B}_t] \leq_{LR} [w_i|\widehat{\mathcal{B}}_t]$ (also established above) implies $[w_i|\mathcal{B}_t, \mathbf{w}_{-i}] \leq_{st} [w_i|\widehat{\mathcal{B}}_t, \mathbf{w}_{-i}]$ (where $\leq_{st}$ denotes stochastic dominance, see p13 Müller and Stoyan 2002). Since we showed $E[\phi(w_i(\mathcal{B}_{t+1}))|\mathcal{B}_t, K_0, \mathbf{w}]$ is nondecreasing in w_i for all K_0, $\mathbf{w}$, and $\mathcal{B}_t$, the inequality in (3.4.15) follows by the definition of stochastic dominance. ■

Fewer past observations encourages auctioning for one extra period with multidimensional Beta-multinomial updating.

Proposition 7. *Suppose the firm updates its priors using the multidimensional Beta-multinomial information structure, and $\mathcal{B}_t$ contains x_0 observations of bidder valuations, while $\widehat{\mathcal{B}}_t$ only contains $y < x_0$ observations. If $w_i(\mathcal{B}_t) = w_i(\widehat{\mathcal{B}}_t) = \bar{w}_i$ for all $i = 1, \ldots, N$ and at time t and sufficient statistic $\mathcal{B}_t$ the firm prefers continuing the auction phase for exactly one additional period (entering at time $t+1$) to entering immediately (entering at time t), the same would also be preferred at time t with sufficient statistic $\widehat{\mathcal{B}}_t$.*

Proof. We first show that for any $s > 0$,

$$E[\max_j \{p_j w_i(\mathcal{B}_{t+s})\}|\mathcal{B}_t] \;\leq\; E[\max_j \{p_j w_i(\widehat{\mathcal{B}}_{t+s})\}|\widehat{\mathcal{B}}_t]. \tag{3.4.16}$$

Let d be the number of bids received during $(t, t+s]$, and let k_i be the number of such bids in the interval $[p_i, p_{i+1})$, where $p_0 \triangleq 0$ and $p_{N+1} \triangleq \infty$. Hence, $d = \sum_{j=0}^{N} k_j$. For shorthand let $\mathbf{k} \triangleq (k_0, \ldots, k_N)$. Set $\bar{w}_0 = 1$ and $\bar{w}_{N+1} = 0$. We establish (3.4.16) by showing that, for all d, $E_{\mathbf{k}}[\max_j\{p_j w_i(\mathcal{B}_{t+s})\}|\mathcal{B}_t, d] \;\leq\; E_{\mathbf{k}}[\max_j\{p_j w_i(\widehat{\mathcal{B}}_{t+s})\}|\widehat{\mathcal{B}}_t, d]$.

$$\begin{aligned}
\text{Let} \qquad j^*(\mathbf{k}, x_0) \;&\triangleq\; \arg\max_j \left\{ \frac{p_j \bar{w}_j x_0 + p_j \sum_{m=j}^{N} k_m}{x_0 + d} \right\}, \\
\text{and let} \qquad g(x) \;&\triangleq\; E_{\mathbf{k}} \left[\frac{p_{j^*(\mathbf{k},x_0)} \bar{w}_{j^*(\mathbf{k},x_0)} x + p_{j^*(\mathbf{k},x_0)} \sum_{m=j^*(\mathbf{k},x_0)}^{N} k_m}{x + d} \right].
\end{aligned}$$

Expanding the expectation we can write

$$g(x) = \frac{d!}{x+d} \sum_{\substack{k_0,\ldots,k_N \\ s.t. \sum_{j=0}^N k_j = d}} \prod_{l=0}^{N} \frac{(\bar{w}_l - \bar{w}_{l+1})^{k_l}}{k_l!} \left[p_{j^*(\mathbf{k},x_0)} \bar{w}_{j^*(\mathbf{k},x_0)} x + p_{j^*(\mathbf{k},x_0)} \sum_{m=j^*(\mathbf{k},x_0)}^{N} k_m \right].$$

For any constants c and d, $\frac{\partial}{\partial x} \frac{p_j \bar{w}_j x + p_j c}{x+d} = \frac{d p_j \bar{w}_j - p_j c}{(x+d)^2}$. Letting

$$h_m(\mathbf{k}) = \begin{cases} p_{j^*(\mathbf{k},x_0)} \bar{w}_{j^*(\mathbf{k},x_0)} - p_{j^*(\mathbf{k},x_0)} & \text{if } m \geq j^*(\mathbf{k}, x_0), \\ p_{j^*(\mathbf{k},x_0)} \bar{w}_{j^*(\mathbf{k},x_0)} & \text{otherwise,} \end{cases}$$

and letting $\mathbf{e}^i$ be the vector of all zeros except for a 1 in i^{th} position, we can write

$$\begin{aligned} \frac{\partial g(x)}{\partial x} &= \frac{d!}{(x+d)^2} \sum_{\substack{k_0,\ldots,k_N \\ s.t. \sum_{j=0}^N k_j = d}} \prod_{l=0}^{N} \frac{(\bar{w}_l - \bar{w}_{l+1})^{k_l}}{k_l!} \sum_{m=0}^{N} k_m h_m(\mathbf{k}), \\ &= \frac{(d-1)!}{(x+d)^2} \sum_{\substack{k_0,\ldots,k_N \\ s.t. \sum_{j=0}^N k_j = d-1}} \prod_{l=0}^{N} \frac{(\bar{w}_l - \bar{w}_{l+1})^{k_l}}{k_l!} \sum_{i=0}^{N} (\bar{w}_i - \bar{w}_{i+1}) \\ &\qquad \times \left[h_i(\mathbf{k} + \mathbf{e}^i) + \sum_{m=0}^{N} k_m h_m(\mathbf{k} + \mathbf{e}^i) \right]. \end{aligned}$$

We now re-write the terms involving the summation over m:

$$\begin{aligned} &\frac{(d-1)!}{(x+d)^2} \sum_{\substack{k_0,\ldots,k_N \\ s.t. \sum_{j=0}^N k_j = d-1}} \prod_{l=0}^{N} \frac{(\bar{w}_l - \bar{w}_{l+1})^{k_l}}{k_l!} \sum_{i=0}^{N} (\bar{w}_i - \bar{w}_{i+1}) \sum_{m=0}^{N} k_m h_m(\mathbf{k} + \mathbf{e}^i) \\ &= \frac{(d-1)!}{(x+d)^2} \sum_{i=0}^{N} \sum_{m=0}^{N} \sum_{\substack{k_0,\ldots,k_N \\ s.t. \sum_{j=0}^N k_j = d-1,\ k_m \geq 1}} \prod_{\substack{l=0,\ldots,N \\ l \neq i,m}} \frac{(\bar{w}_l - \bar{w}_{l+1})^{k_l}}{k_l!} \\ &\quad \cdot \frac{(\bar{w}_i - \bar{w}_{i+1})^{k_i+1}}{(k_i+1)!} \frac{(\bar{w}_m - \bar{w}_{m+1})^{k_m-1}}{(k_m-1)!} (k_i+1) h_m(\mathbf{k} + \mathbf{e}^i)(\bar{w}_m - \bar{w}_{m+1}), \end{aligned}$$

where we have used the fact that the terms involving $k_m = 0$ are zeroed out. Substituting $\tilde{k}_i = k_i + 1$, $\tilde{k}_m = k_m - 1$, $\tilde{k}_j = k_j$ for $j \neq i, m$, we can re-write the terms as

$$\begin{aligned}
&\frac{(d-1)!}{(x+d)^2} \sum_{i=0}^{N} \sum_{m=0}^{N} \sum_{\substack{\tilde{k}_0,\ldots,\tilde{k}_N \\ s.t. \sum_{j=0}^{N} \tilde{k}_j = d-1,\ \tilde{k}_i \geq 1}} \prod_{l=0}^{N} \frac{(\bar{w}_l - \bar{w}_{l+1})^{\tilde{k}_l}}{\tilde{k}_l!} \tilde{k}_i h_m(\tilde{\mathbf{k}} + \mathbf{e}^m)(\bar{w}_m - \bar{w}_{m+1}) \\
&= \frac{(d-1)!}{(x+d)^2} \sum_{m=0}^{N} \sum_{\substack{\tilde{k}_0,\ldots,\tilde{k}_N \\ s.t. \sum_{j=0}^{N} \tilde{k}_j = d-1}} \prod_{l=0}^{N} \frac{(\bar{w}_l - \bar{w}_{l+1})^{\tilde{k}_l}}{\tilde{k}_l!} (d-1) h_m(\tilde{\mathbf{k}} + \mathbf{e}^m)(\bar{w}_m - \bar{w}_{m+1}),
\end{aligned}$$

where we have used the fact that $\sum_{i=0}^{N} \tilde{k}_k = d - 1$ and terms for which $\tilde{k}_i = 0$ zero out. Hence,

$$\begin{aligned}
\frac{\partial g(x)}{\partial x} &= \frac{(d-1)!}{(x+d)^2} \sum_{\substack{k_0,\ldots,k_N \\ s.t. \sum_{j=0}^{N} k_j = d-1}} \prod_{l=0}^{N} \frac{(\bar{w}_l - \bar{w}_{l+1})^{k_l}}{k_l!} \sum_{i=0}^{N} (\bar{w}_i - \bar{w}_{i+1}) \\
&\qquad \times \left[h_i(\mathbf{k} + \mathbf{e}^i) + (d-1) h_i(\mathbf{k} + \mathbf{e}^i) \right] \\
&= \frac{d!}{(x+d)^2} \sum_{\substack{k_0,\ldots,k_N \\ s.t. \sum_{j=0}^{N} k_j = d-1}} \prod_{l=0}^{N} \frac{(\bar{w}_l - \bar{w}_{l+1})^{k_l}}{k_l!} \sum_{i=0}^{N} (\bar{w}_i - \bar{w}_{i+1}) h_i(\mathbf{k} + \mathbf{e}^i).
\end{aligned}$$

We next show that $\sum_{i=0}^{N} (\bar{w}_i - \bar{w}_{i+1}) h_i(\mathbf{k} + \mathbf{e}^i) \leq 0$ for all fixed $\mathbf{k}$ such that $\sum_{j=0}^{N} k_j = d - 1$ (hence $\partial g / \partial x \leq 0$). For such a fixed $\mathbf{k}$, define

$$j_q \triangleq \arg\max_j \left\{ \frac{p_j \bar{w}_j x_0 + p_j \sum_{m=j}^{N} k_m}{x_0 + d - 1} \right\}.$$

Price p_{j_q} may or may not remain optimal once the d^{th} arrival is considered. If the d^{th} arrival has a valuation below p_1, the optimal choice of price will not change from p_{j_q}. Thus, $h_0(\mathbf{k} + \mathbf{e}^0) = p_{j_q} \bar{w}_{j_q}$. If the d^{th} arrival has a valuation above p_1 but below p_2, the optimal price will either change to p_1 or remain at p_{j_q}. If the d^{th} arrival has a valuation above p_2 but below p_3, the optimal price will either change to p_2 or remain at p_1 or p_{j_q} (whichever price was optimal for an arrival between p_1 and

p_2). Continuing in this manner, define the indices, $j_1, \ldots, j_{q-1}$, as the $q-1$, $q \geq 1$, indices for which the optimal price changes for arrivals with valuations less than p_{j_q}. Define the indices, $j_{q+1}, \ldots, j_{q+r}$, as the r, $r \geq 0$, times the optimal price changes for arrivals with valuations greater than p_{j_q}. Let $l^* = \arg\max_{l=q,\ldots,q+r}\{p_{j_l}\bar{w}_{j_l} - p_{j_l}\}$, and set $\bar{w}_{j_{r+q+1}} = 0$. We can then write

$$\begin{aligned}
\sum_{i=0}^{N}(\bar{w}_i - \bar{w}_{i+1})h_i(\mathbf{k}+\mathbf{e}^i) &= (1-\bar{w}_{j_1})p_{j_q}\bar{w}_{j_q} + \sum_{l=1}^{q+r}(\bar{w}_{j_l} - \bar{w}_{j_l+1})(p_{j_l}\bar{w}_{j_l} - p_{j_l}) \\
&\leq (1-\bar{w}_{j_1})p_{j_q}\bar{w}_{j_q} + \sum_{l=1}^{q-1}(\bar{w}_{j_l} - \bar{w}_{j_l+1})(p_{j_l}\bar{w}_{j_l} - p_{j_l}) + \bar{w}_{j_q}(p_{j_{l^*}}\bar{w}_{j_{l^*}} - p_{j_{l^*}}) \\
&\leq (1-\bar{w}_{j_1})p_{j_{l^*}}\bar{w}_{j_{l^*}} + \sum_{l=1}^{q-1}(\bar{w}_{j_l} - \bar{w}_{j_l+1})(p_{j_l}\bar{w}_{j_l} - p_{j_l}) + \bar{w}_{j_q}(p_{j_{l^*}}\bar{w}_{j_{l^*}} - p_{j_{l^*}}), \\
&= (\bar{w}_{j_q} - \bar{w}_{j_1})p_{j_{l^*}}\bar{w}_{j_{l^*}} + \sum_{l=1}^{q-1}(\bar{w}_{j_l} - \bar{w}_{j_l+1})(p_{j_l}\bar{w}_{j_l} - p_{j_l}) + p_{j_{l^*}}(\bar{w}_{j_{l^*}} - \bar{w}_{j_q}) \leq 0.
\end{aligned}$$

The second inequality follows since $l^* \geq q$ implies $p_{j_{l^*}} \geq p_{j_q}$ and hence $p_{j_{l^*}}\bar{w}_{j_{l^*}} \geq p_{j_q}\bar{w}_{j_q}$. The final inequality follows since $\bar{w}_i \geq \bar{w}_j$ for $i \leq j$, and $\bar{w}_i \leq 1$ for all i.

Now, having established that $\partial g/\partial x \leq 0$, we can write

$$\begin{aligned}
E_{\mathbf{k}}[\max_j\{p_j w_j(\mathcal{B}_{t+s})\}|\mathcal{B}_t, d] &= g(x_0) \leq g(y) \\
&= E_{\mathbf{k}}\left[\frac{p_{j^*(\mathbf{k},x_0)}\bar{w}_{j^*(\mathbf{k},x_0)}y + p_{j^*(\mathbf{k},x_0)}\sum_{m=j^*(\mathbf{k},x_0)}^{N} k_m}{y+d}\right] \\
&\leq E_{\mathbf{k}}\left[\max_j\left\{\frac{p_j\bar{w}_j y + p_j\sum_{m=j}^{N} k_m}{y+d}\right\}\right] \\
&= E_{\mathbf{k}}[\max_j\{p_j w_j(\widehat{\mathcal{B}}_{t+s})\}|\widehat{\mathcal{B}}_t, d].
\end{aligned}$$

Because the above holds for all d, we have established (3.4.16), from which the proposition follows easily. ■

Proof of Proposition 2

We begin with part 1. Suppose that at time $t-1$ and sufficient statistic $\mathcal{B}_{t-1} = \mathcal{B}$, stopping the auction phase and entering the mass market is optimal. We show that stopping would also be optimal were the time instead t. Let $t + K$ be the optimal stopping time over horizon $[t, T]$ for a firm with sufficient statistic $\mathcal{B}$ at time t. If during horizon $[t-1, T]$ the firm pretends to have started from time t instead of $t-1$, it can do no better than if it optimizes its stopping time relative to its true starting point of $t-1$. Thus if stopping is optimal at time $t-1$, then

$$\sum_{k=1}^{T-t} \Pr(K = k)\delta^k E_{\mathcal{B}_{t-1+k}|K=k}[\max_i\{p_i w_i(\mathcal{B}_{t-1+k})\}|\mathcal{B}_{t-1} = \mathcal{B}]M(t-1+k)$$
$$\leq \max_i\{p_i w_i(\mathcal{B})\}M(t-1).$$

Dividing both sides by $M(t-1)$ yields

$$\sum_{k=1}^{T-t} \Pr(K = k)\delta^k E_{\mathcal{B}_{t-1+k}|K=k}[\max_i\{p_i w_i(\mathcal{B}_{t-1+k})\}|\mathcal{B}_{t-1} = \mathcal{B}]\frac{M(t-1+k)}{M(t-1)}$$
$$\leq \max_i\{p_i w_i(\mathcal{B})\}.$$

For now, suppose that $M(t-1+k)/M(t-1) \geq M(t+k)/M(t)$ for all $k \in \{1, \ldots, T-t\}$. Thus,

$$\sum_{k=1}^{T-t} \Pr(K = k)\delta^k E_{\mathcal{B}_{t-1+k}|K=k}[\max_i\{p_i w_i(\mathcal{B}_{t-1+k})\}|\mathcal{B}_{t-1} = \mathcal{B}]\frac{M(t+k)}{M(t)}$$
$$\leq \max_i\{p_i w_i(\mathcal{B})\}.$$

Since the data updating process is assumed to be stationary, we can write

$$\sum_{k=1}^{T-t} \Pr(K=k)\delta^k E_{\mathcal{B}_{t+k}|K=k}[\max_i\{p_i w_i(\mathcal{B}_{t+k})\}|\mathcal{B}_t = \mathcal{B}]\frac{M(t+k)}{M(t)} \qquad (3.4.17)$$
$$\leq \max_i\{p_i w_i(\mathcal{B})\}.$$

The LHS of (3.4.17) is the payoff (divided by $M(t)$) of continuing auctioning at time t when following optimal stopping policy $t+K$. Thus, (3.4.17) implies that stopping the auction phase is optimal at time t and sufficient statistic $\mathcal{B}_t = \mathcal{B}$, as long as we can show $M(t-1+k)/M(t-1) \geq M(t+k)/M(t)$. Clearly the inequality holds at $k = T-t$, since $M(T) = 0$. For $k \in \{1, \ldots, T-t-1\}$ it is sufficient to show that

$$\frac{\partial}{\partial s}\left(\frac{M(s+k)}{M(s)}\right) \leq 0$$

for $s \in [t-1, t]$. Since $M(s+k) > 0$ for $k = 1, \ldots, T-t-1$, this is equivalent to

$$\frac{\frac{\partial}{\partial s}(M(s+k))}{M(s+k)} \leq \frac{\frac{\partial}{\partial s}(M(s))}{M(s)},$$

which holds if $M(\cdot)$ is log-concave over $[t-1, T-1]$.

Proof of part 2 is analogous to that of part 1. Suppose that at time $t-1$ and sufficient statistic $\mathcal{B}_{t-1} = \mathcal{B}$, continuing the auction phase is optimal. We show that continuing the auction phase would also be optimal were the time instead t. Let $t-1+K$ be the optimal stopping time if following an optimal policy from time $t-1$. Since by assumption $t-1+K \leq T-2$ with probability one, if continuing is optimal at time $t-1$, then

$$\sum_{k=1}^{T-t-1} \Pr(K=k)\delta^k E_{\mathcal{B}_{t-1+k}|K=k}[\max_i\{p_i w_i(\mathcal{B}_{t-1+k})\}|\mathcal{B}_{t-1} = \mathcal{B}]\frac{M(t-1+k)}{M(t-1)}$$
$$\geq \max_i\{p_i w_i(\mathcal{B})\}.$$

For now, suppose that $M(t-1+k)/M(t-1) \leq M(t+k)/M(t)$ for all $k \in \{1, \ldots, T-t-1\}$. Thus,

$$\sum_{k=1}^{T-t-1} \Pr(K=k)\delta^k E_{\mathcal{B}_{t-1+k}|K=k}[\max_i\{p_i w_i(\mathcal{B}_{t-1+k})\}|\mathcal{B}_{t-1}=\mathcal{B}]\frac{M(t+k)}{M(t)} \geq \max_i\{p_i w_i(\mathcal{B})\}.$$

Since the data updating process is assumed to be stationary, we can write

$$\sum_{k=1}^{T-t-1} \Pr(K=k)\delta^k E_{\mathcal{B}_{t+k}|K=k}[\max_i\{p_i w_i(\mathcal{B}_{t+k})\}|\mathcal{B}_t=\mathcal{B}]\frac{M(t+k)}{M(t)} \geq \max_i\{p_i w_i(\mathcal{B})\}. \tag{3.4.18}$$

The LHS of (3.4.18) is the payoff (divided by $M(t)$) of continuing auctioning at time t and stopping at $t+K$ per the optimal policy for the firm starting from time $t-1$. When continuing auctioning from time t, the true optimal stopping policy in $[t+1, T]$ performs at least as well as when following the stopping time policy $t+K$. Thus, (3.4.18) implies that continuing the auction phase is optimal at time t and sufficient statistic $\mathcal{B}_t = \mathcal{B}$, as long as we can show $M(t-1+k)/M(t-1) \leq M(t+k)/M(t)$. It is sufficient to show that

$$\frac{\partial}{\partial s}\left(\frac{M(s+k)}{M(s)}\right) \geq 0$$

for all $k \in \{1, \ldots, T-t-1\}$, $s \in [t-1, t]$. Since $M(s+k) > 0$ for $k \in \{1, \ldots, T-t-1\}$, this is equivalent to

$$\frac{\frac{\partial}{\partial s}(M(s+k))}{M(s+k)} \geq \frac{\frac{\partial}{\partial s}(M(s))}{M(s)},$$

which holds if $M(\cdot)$ is log-convex over $[t-1, T-1]$. ■

Proof of Proposition 3

We first show that $\partial/\partial x(M(s)/M(t))\big|_x \geq 0$ for all $t \leq s \leq T$, $x \in I$, implies the stopping region shrinks as the market size parameter, x, increases within I. Let $x_1, x_2 \in I$. Suppose that at time t, with sufficient statistic $\mathcal{B}_t$ and market size parameter x_1, continuing the auction phase is optimal. We show that continuing the auction phase would also be optimal were the market size parameter $x_2 \geq x_1$. Let $M_k(t)$ be the market size at time t with parameter x_k, and S_k be the optimal stopping time if following an optimal policy as if the market size function is M_k, $k \in \{1, 2\}$. If continuing is optimal with parameter x_1, then

$$\sum_{s=t+1}^{T} \Pr(S_1 = s)\delta^{s-t} E_{\mathcal{B}_s|S_1=s}[\max_i\{p_i w_i(\mathcal{B}_s)\}|\mathcal{B}_t]\frac{M_1(s)}{M_1(t)} \;\geq\; \max_i\{p_i w_i(\mathcal{B}_t)\}.$$

When the market parameter is x_2, the optimal stopping policy, $S_2 \in [t+1, T]$, performs at least as well as when following the stopping time policy, S_1, which is optimal under market parameter x_1. By assumption we have $M_1(s)/M_1(t) \leq M_2(s)/M_2(t)$. Thus, when continuing the auction phase is optimal at time t, sufficient statistic $\mathcal{B}_t$, and market size parameter x_1 we have

$$\begin{aligned}
&\sum_{s=t+1}^{T} \Pr(S_2 = s)\delta^{s-t} E_{\mathcal{B}_s|S_2=s}[\max_i\{p_i w_i(\mathcal{B}_s)\}|\mathcal{B}_t]\frac{M_2(s)}{M_2(t)} \\
&\geq \sum_{s=t+1}^{T} \Pr(S_1 = s)\delta^{s-t} E_{\mathcal{B}_s|S_1=s}[\max_i\{p_i w_i(\mathcal{B}_s)\}|\mathcal{B}_t]\frac{M_2(s)}{M_2(t)} \qquad (3.4.19)\\
&\geq \max_i\{p_i w_i(\mathcal{B}_t)\},
\end{aligned}$$

and continuing the auction phase is optimal at time t, sufficient statistic $\mathcal{B}_t$, and market size parameter x_2.

The proof of the expanding stopping regions case is analogous. With parameter x_1, if stopping at t with $\mathcal{B}_t$ is optimal then continuing and following stopping time S_2

is worse than stopping. This and the assumption that $M_1(s)/M_1(t) \geq M_2(s)/M_2(t)$ can be used to show that stopping is also optimal with parameter $x_2 \geq x_1$. We omit the details for brevity. Combining the arguments for both of the above cases implies that the stopping regions do not change if $\partial/\partial x(M(s)/M(t)) = 0$ for $s \geq t$. ■

Proof of Corollary 3

Differentiating the natural log of (3.3.6) twice with respect to t is easily shown to always be non-negative, immediately yielding the result. ■

Proof of Corollary 4

The result follows by showing that $M(s)/M(t)$ is monotonically increasing in α and that Proposition 3 applies. The result holds trivially for $s = T$. For $t \leq s < T$,

$$\frac{\partial}{\partial t}\left(\frac{\frac{\partial}{\partial \alpha}M(t)}{M(t)}\right) \geq 0 \Rightarrow \frac{\frac{\partial}{\partial \alpha}M(s)}{M(s)} \geq \frac{\frac{\partial}{\partial \alpha}M(t)}{M(t)} \Rightarrow \frac{\partial}{\partial \alpha}\left(\frac{M(s)}{M(t)}\right) \geq 0.$$

Accordingly, we will show that

$$\frac{\partial}{\partial t}\left(\frac{\frac{\partial}{\partial \alpha}M(t)}{M(t)}\right) \geq 0$$

for all $\alpha, \beta > 0$ and all $t \in [0, T]$. Let $\gamma \triangleq (\alpha + \beta)(T - t)$. The derivative,

$$\frac{\partial}{\partial t}\left(\frac{\frac{\partial}{\partial \alpha}M(t)}{M(t)}\right) = \frac{(\alpha+\beta)e^{-\gamma}\Big(e^{-\gamma}(\alpha - 3\beta) + \beta e^{-2\gamma}(\gamma+2) + \alpha\gamma - \alpha + \beta\Big)}{(\alpha + \beta e^{-\gamma})^2(1 - e^{-\gamma})^2},$$

will be non-negative if and only if

$$D(t) \triangleq e^{-\gamma}(\alpha - 3\beta) + \beta e^{-2\gamma}(\gamma + 2) + \alpha\gamma - \alpha + \beta \geq 0.$$

Since $D(T) = 0$ for all α, β, if we can show that $D(t)$ is decreasing on $[0, T]$, it will follow that $D(t)$ must be non-negative. Differentiation yeilds

$$\frac{\partial D}{\partial t} = (\alpha + \beta)\Big(e^{-\gamma}(\alpha - 3\beta) + \beta e^{-2\gamma}(2\gamma + 3) - \alpha\Big).$$

$D(t)$ will be decreasing if and only if

$$F(t) \triangleq e^{-\gamma}(\alpha - 3\beta) + \beta e^{-2\gamma}(2\gamma + 3) - \alpha \leq 0.$$

Since $F(T) = 0$, $F(t)$ will be non-positive if either (i) $F(t)$ is increasing on $[0, T]$, or (ii) $F(t)$ is unimodal on $[0, T]$ (decreasing then increasing) with $F(0) \leq 0$. Taking the derivative of $F(t)$,

$$\frac{\partial F}{\partial t} = (\alpha + \beta)e^{-\gamma}\Big(4\beta e^{-\gamma}(1 + \gamma) + \alpha - 3\beta\Big).$$

Define

$$G(t) \triangleq 4\beta e^{-\gamma}(1 + \gamma) + \alpha - 3\beta, \quad \text{with} \quad \frac{\partial G}{\partial t} = 4\beta(\alpha + \beta)\gamma e^{-\gamma} \geq 0.$$

The derivative of $G(t)$ is non-negative, which implies that $G(t)$ is increasing on $[0, T]$. This also implies that the derivative of $F(t)$ can change signs at most one time. Also, since $G(T) = \alpha + \beta > 0$, $F(t)$ is increasing at $t = T$. Thus, $F(t)$ is either (i) increasing on $[0, T]$ or (ii) decreasing then increasing on $[0, T]$.

Case 1. $G(0) \geq 0$. Since $G(t)$ is increasing, $G(t) \geq 0$ for all $t \in [0, T]$, which implies that $F(t)$ is increasing for all $t \in [0, T]$. Thus, $F(t) \leq 0$ and $D(t) \geq 0$ for all $t \in [0, T]$.

Case 2. $G(0) < 0$. Since $G(t)$ is increasing and $G(T) > 0$, we know that $F(t)$ is unimodal (decreasing then increasing) on $[0, T]$. To finish the proof, it is necessary to show that $F(0) < 0$ when $G(0) < 0$. To this end, note that $e^{-\gamma}G(t) < 0 \iff$

$G(t) < 0$. Furthermore,

$$
\begin{aligned}
F(t) - e^{-\gamma}G(t) &= e^{-\gamma}(\alpha - 3\beta) + \beta e^{-2\gamma}(2\gamma + 3) - \alpha - 4\beta e^{-2\gamma}(1 + \gamma) - e^{-\gamma}(\alpha - 3\beta), \\
&= -\beta e^{-2\gamma}(2\gamma + 1) - \alpha < 0.
\end{aligned}
$$

Thus, if $G(0) < 0$, then $F(0) < 0$ and $F(t)$ is decreasing then increasing on $[0, T]$, implying that $F(t) \leq 0$ and $D(t) \geq 0$ for all $t \in [0, T]$. ■

Proof of Corollary 5

The proof is similar in spirit to that of Corollary 4. First, note that $\partial/\partial\beta(M(s)/M(t)) \leq 0$ holds trivially if $s = T$. For fixed β, we show there exists a $\hat{t}(\beta) < T$ such that

$$
\frac{\partial}{\partial t}\left(\frac{\frac{\partial}{\partial\beta}M(t)}{M(t)}\right) < 0
$$

for all $t \in [\hat{t}(\beta), T)$. For readability, where convenient we will suppress the argument β when writing $\hat{t}$. Let $\gamma \triangleq (\alpha + \beta)(T - t)$. We have

$$
\frac{\partial}{\partial t}\left(\frac{\frac{\partial}{\partial\beta}M(t)}{M(t)}\right) < 0 \iff D(t) \triangleq (-3\alpha + \beta)e^{-\gamma} - (\beta\gamma - \alpha + \beta)e^{-2\gamma} - \alpha(\gamma - 2) > 0.
$$

It is easy to verify that $D(T) = 0$. We show there exists $\hat{t} < T$ such that $\partial D(t)/\partial t < 0$ for all $t \in [\hat{t}, T)$. First,

$$
\frac{\partial D(t)}{\partial t} > 0 \iff F(t) \triangleq (-3\alpha + \beta)e^{-\gamma} - (2\beta\gamma - 2\alpha + \beta)e^{-2\gamma} + \alpha > 0 \quad \text{and}
$$

$$
\frac{\partial F(t)}{\partial t} > 0 \iff G(t) \triangleq -3\alpha + \beta - 4(\beta\gamma - \alpha)e^{-\gamma} > 0.
$$

$G(T) = \alpha + \beta > 0$ implies $F(t)$ is strictly increasing at $t = T$. Since $F(T) = 0$, we have that there exists $\hat{t} < T$ such that $F(t) < 0$ for all $t \in [\hat{t}, T)$. Hence, $\partial D(t)/\partial t < 0$

for all $t \in [\hat{t}, T)$ and $D(t) > 0$ for all $t \in [\hat{t}, T)$.

Next, we show that $\hat{t} = 0$ for α sufficiently small. In particular, for α sufficiently small,

$$\frac{\partial}{\partial t}\left(\frac{\frac{\partial}{\partial \beta}M(t)}{M(t)}\right) < 0 \qquad \text{for } 0 \leq t \leq T-1.$$

It is easy to check that if $\alpha = 0$, $D(t) > 0 \iff \beta(T-t) > \ln(\beta(T-t)+1)$, which holds for all $t \leq T-1$ and $\beta > 0$. For the moment, we find it convenient to think of D as a function of both t and α. Since D is continuous in t and α, D is uniformly continuous over compact domain $(t, \alpha) \in [0, T-1] \times [0, 1]$, where, without loss of generality, for compactness we have chosen 1 as an upper bound on α. Thus, there exists $\alpha_0 > 0$ such that $0 < \alpha < \alpha_0$ implies $D(t) > 0$ for all $t \in [0, T-1]$.

Fixing α, if $\hat{t}(\beta_0) \leq T-1$, setting $t_0 = \hat{t}(\beta_0)$ implies $\partial/\partial\beta(M(s)/M(t))\big|_{\beta=\beta_0} < 0$ for $t_0 \leq t \leq s \leq T-1$. Since D is continuous in t and β, $D(t)$ is uniformly continuous over compact domain $(t, \beta) \in [t_0, T-1] \times [0, \beta_0+1]$, where, without loss of generality, we have chosen $\beta_0 + 1$ as an upper bound on the β. Thus, there exists $0 < \delta < 1$ such that $\beta \in (\beta_0 - \delta, \beta_0 + \delta)$ implies $D(t) > 0$ for all $t \in [0, T-1]$. Finally, if $\hat{t}(\beta_0) > T-1$, set $t_0 = T$ (the trivial case). Applying Proposition 3 completes the proof. ■

Proof of Proposition 4

Without loss of generality, we will assume that the fixed cost c is paid at the end of each auctioning period, that is, (3.2.3) becomes

$$J_t(\mathcal{B}_t) = \max\Big[\max_j\{p_j w_j(\mathcal{B}_t)M(t)\}, -\delta c + \delta E[J_{t+1}(\mathcal{B}_{t+1})|\mathcal{B}_t]\Big]. \tag{3.4.20}$$

The proof of Proposition 1 showed that for all $\mathcal{B}_t \in \Omega_t$, $p_i w_i(\mathcal{B}_t)M(t) - \delta E[J_{t+1}(\mathcal{B}_{t+1})|\mathcal{B}_t]$ is nondecreasing in $w_i(\mathcal{B}_t)$ for all t and i. Since δc is just a constant, the same proof can be used to show that $p_i w_i(\mathcal{B}_t)M(t) + \delta c - \delta E[J_{t+1}(\mathcal{B}_{t+1})|\mathcal{B}_t]$ is nondecreasing in $w_i(\mathcal{B}_t)$ for all t and i, and the result follows.

To show that part 1 of Proposition 2 holds, we use an argument very similar to the original proof of part 1 of Proposition 2, whose notation we reuse here. Suppose that at time $t-1$ and sufficient statistic $\mathcal{B}_{t-1} = \mathcal{B}$, stopping the auction phase and entering the mass market is optimal. We show that stopping would also be optimal were the time instead t. Let $t+K$ be the optimal stopping time over horizon $[t,T]$ for a firm with sufficient statistic $\mathcal{B}$ at time t. If during horizon $[t-1,T]$ the firm pretends to have started from time t instead of $t-1$, it can do no better than if it optimizes its stopping time relative to its true starting point of $t-1$. Thus if stopping is optimal at time $t-1$, then

$$\sum_{k=1}^{T-t} \Pr(K=k)\delta^k E_{\mathcal{B}_{t-1+k}|K=k}[\max_i\{p_i w_i(\mathcal{B}_{t-1+k})\}|\mathcal{B}_{t-1}=\mathcal{B}]\frac{M(t-1+k)}{M(t-1)}$$
$$-\sum_{k=1}^{T-t}\Pr(K\geq k)\delta^k\frac{c}{M(t-1)} \;\leq\; \max_i\{p_i w_i(\mathcal{B})\}.$$

Log-concavity of M over $[t-1,T-1]$ implies that $M(t-1+k)/M(t-1) \geq M(t+k)/M(t)$ for all $k=1\ldots T-t$. Together with $M(t-1)\geq M(t)$ and stationarity of the data updating process, this implies

$$\sum_{k=1}^{T-t} \Pr(K=k)\delta^k E_{\mathcal{B}_{t+k}|K=k}[\max_i\{p_i w_i(\mathcal{B}_{t+k})\}|\mathcal{B}_{t}=\mathcal{B}]\frac{M(t+k)}{M(t)}$$
$$-\sum_{k=1}^{T-t}\Pr(K\geq k)\delta^k\frac{c}{M(t)} \;\leq\; \max_i\{p_i w_i(\mathcal{B})\}. \quad (3.4.21)$$

The LHS of the inequality above is the payoff of continuing auctioning at time t when following optimal stopping policy $t+K$. Thus, stopping the auction phase is optimal at time t and sufficient statistic $\mathcal{B}_t = \mathcal{B}$. ■

Proof of Proposition 5

The argument is similar in spirit to that used in the proof of Proposition 3, whose notation we reuse here. We first show that $\partial/\partial x(M(s)/M(t))\big|_x \geq 0$ and $\partial/\partial x M(t)\big|_x \geq 0$ for all $s \geq t$, $x \in I$, implies the stopping region shrinks with market size parameter, x, for $x \in I$. Let $x_1, x_2 \in I$, and $x_1 \leq x_2$. Suppose that at time t, with sufficient statistic $\mathcal{B}_t$ and market size parameter x_1, continuing the auction phase is optimal. We show that continuing the auction phase would also be optimal were the market size parameter $x_2 \geq x_1$. Let $M_k(t)$ be the market size at time t with parameter x_k, and S_k be the optimal stopping time if following an optimal policy as if the market size function is M_k, $k \in \{1, 2\}$. If continuing is optimal with parameter x_1, then

$$\sum_{s=t+1}^{T} \Pr(S_1 = s)\delta^{s-t} E_{\mathcal{B}_s|S_1=s}[\max_i\{p_i w_i(\mathcal{B}_s)\}|\mathcal{B}_t]\frac{M_1(s)}{M_1(t)} - \sum_{s=t+1}^{T} \Pr(S_1 \geq s)\delta^{s-t}\frac{c}{M_1(t)}$$
$$\geq \max_i\{p_i w_i(\mathcal{B}_t)\}.$$

Note that this is the same equation as derived in the proof of Proposition 3, except for the term involving c, the cost of auctioning for another period. By assumption we have $M_1(s)/M_1(t) \leq M_2(s)/M_2(t)$, and $M_1(t) \leq M_2(t)$ implies $c/M_1(t) \geq c/M_2(t)$. Thus,

$$\sum_{s=t+1}^{T} \Pr(S_1 = s)\delta^{s-t} E_{\mathcal{B}_s|S_1=s}[\max_i\{p_i w_i(\mathcal{B}_s)\}|\mathcal{B}_t]\frac{M_2(s)}{M_2(t)} - \sum_{s=t+1}^{T} \Pr(S_1 \geq s)\delta^{s-t}\frac{c}{M_2(t)}$$
$$\geq \max_i\{p_i w_i(\mathcal{B}_t)\}. \quad (3.4.22)$$

When the market parameter is x_2, the true optimal stopping policy S_2 in $[t+1, T]$ performs at least as well as when following the stopping time policy S_1 that would be optimal were the market parameter instead equal to x_1. Thus, (3.4.22) implies that continuing the auction phase is optimal at time t, sufficient statistic $\mathcal{B}_t$, and market size parameter x_2. Similar reasoning applies for the other cases of the proposition. ■

Proof of Proposition 6

The argument follows immediately from noting that the LHS of (3.4.21) decreases in c. ■

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Chapter 4

On the Theory Auctions with a Fixed Price Purchasing Option for Losing Bidders

4.1 Introduction

Consider the following situation. A firm holds an auction for a single unit of a new product. Bidders in the auction know that the product will be offered for sale at a fixed price at a retail outlet in the near future. Intuitively, the option of purchasing the product at a fixed price after losing the auction will cause the participants in the auction to lower their bids by some amount.

In this chapter we present four different auction mechanisms to address this situation. These mechanisms are not necessarily optimal in the sense of maximizing auction revenues, but they can be used to effectively gather willingness to pay information from consumers. Presumably, we would like to construct auction mechanisms in which the conversion from bid data to the distribution of consumer valuations is trivial.

We allow the auction to take the form of a first-price or a second-price auction. Additionally, after the fixed price is set, the firm may, or may not, choose to offer a rebate to the winner of the auction if the price paid at auction exceeds the fixed price set by the firm. We refer to these four cases as first-price auction without rebate, first-price auction with rebate, second-price auction without rebate, and second-price auction with rebate. In this chapter, we find the symmetric equilibrium bidding strategies for each of these mechanisms. Of particular interest, we find that the second-price auction with rebate induces bidders to truthfully reveal their valuations during the auction.

In the preceding chapter, we used the data from auctions to update our beliefs of consumers' valuations for a new product. An updating scheme must account for the type of signals which it processes. The signals provided by the auction will depend on the auction format, bidder beliefs, and bidder behavior. In the recent auction marketing research literature, Harrison et al. (2004) pointed out the need to account for data censoring due to issues such as bidder beliefs about outside purchasing options (e.g., bidders knowing the item will eventually be sold via fixed price) or affiliated bidder beliefs about the outside option or product quality (which can influence the bidding in an open-bid, e.g., English format). One could attempt to address these issues with an elaborate updating scheme that carefully reverse engineers consumers' true willingness to pay from their observed bids, and indeed sophisticated reverse engineering has been done in studies on field data from auctions featuring confounding factors such as publicly announced reserve prices that censor the available observations (Paarsch 1997). However, another option, available if the researcher controls the auction design, is to design an auction format which makes the willingness-to-pay distribution more easily recoverable from the observed bids. This motivates the second-price auction with rebate mechanism discussed in this chapter, which is designed to elicit truthful bidding from consumers. Truthful bidding simplifies the firm's belief updating process by eliminating the need for it to reverse engineer bid data to recover valuations. This auction format is a first step at auction design for the express purpose of learning demand to inform a fixed price that will later be available to consumers. To our knowledge it is the first auction mechanism for market research designed to take into account a future fixed price purchasing opportunity.

While the marketing literature has not considered auctions followed by an outside purchase opportunity, there has been considerable treatment in the field of finance with respect to the underpricing of initial public offerings (IPOs). When a firm sells new shares, traditionally, it hires an underwriting company to determine the offering price. As a form of market research during the book building process, the underwriting firm gathers indications of interest from institutional investors, which will be used to inform the asking price. Benveniste and Spindt (1989) study a mechanism in which investors indicate their interest in an IPO (bids), after which the underwriting firm determines an optimal offer price and allocation schedule based on the bids received. Instead of using book building to sell the IPO, some firms have chosen to use auctions for this purpose, see Biais et al. (2002) and Biais and Faugeron-Crouzet (2002).

The analytical work in this chapter builds upon the theory of first-price and second-price auctions without the option to purchase the item at a future date. For early papers on bidding strategies without a secondary purchase opportunity, see Vickrey (1961), Myerson (1981), Riley and Samuelson (1981), and Milgrom and Weber (1982). For a concise exposition of these works, see Krishna (2002).

The chapter is organized in the following manner. Assumptions underlying our model are provided in §4.2. The equilibrium bidding strategies for the different auction mechanisms are found in §4.3, §4.4, §4.5, and §4.6. All proofs are contained within the text. Concluding remarks are made in §4.7.

4.2 Model Assumptions

We apply the following behavioral assumptions for the equilibrium analysis of this section. These assumptions help formalize our example of how the bid-to-valuation inversion might be aided by auction design. We assume that individual consumers are risk-neutral, have use for a single unit, and if they fail to purchase via auction they will purchase at the fixed price provided it does not exceed their valuation. Consumers are patient, that is, indifferent between receiving the good at auction or purchasing it later at the fixed price provided the payment is the same (i.e., there is no discounting). Consumers who arrive to the auction account for the fact that the item will eventually be available to them at a fixed price. However, they view the future fixed price as exogenous, an assumption best made when many bidders participate in the auction phase, thereby drowning out any one particular bid's influence on the firm's pricing decision. Finally, we assume that the underlying willingness-to-pay distribution is the same for all consumers. This assumption means that the firm can recover mass-market willingness to pay directly from observed bids.

More formally, valuations are independent and identically distributed with cumulative density function $G_0(\cdot)$ on support $[0, \omega]$. Each consumer wants only one item. There is an auction in which the high bidder wins the object. If a bidder does not win the auction, she may purchase the object at a later date for some unknown price, T, where T has distribution $H(\cdot)$ on support $[0, \omega]$. The distribution of the second highest valuation, Y_1, is given by $G(\cdot)$. That is, $G(x) = G_0(x)^{N-1}$ when there are N bidders. Let $\beta(x)$ denote the strictly increasing equilibrium bidding strategy of a bidder with valuation x. A bidder wins the auction with bid, b, if $b \geq \beta(Y_1)$, or equivalently, $Y_1 \leq \beta^{-1}(b)$. Let $\Pi(b, x)$ denote the expected payoff to a bidder with valuation x when bidding b.

The symmetric equilibrium bidding strategies are now found for the following four cases: second-price auction without rebate (**II**); first-price auction without rebate (**I**); first-price auction with rebate ($\mathbf{I_R}$); and second-price auction with rebate ($\mathbf{II_R}$). The cases are presented in this order since an important result from the second-price auction without rebate case is necessary for the analysis of the first-price auction mechanisms. The 'rebate' simply means if a bidder wins the auction and pays more than the fixed price which is determined at a later date, the firm will refund the difference. All auctions are assumed to be sealed bid.

4.3 Second-Price Auction without Rebate

First, consider the common second-price auction format. In this case, no rebate is given to the winning bidder if the fixed price is less than the price paid at auction. We shown that consumers shade their bid when they know the item will be available at a later date for an unknown price, T.

Theorem 1. *In a second-price auction without rebate, a symmetric equilibrium bidding strategy is*

$$\beta^{\mathbf{II}}(x) = x - \int_0^x (x-t)h(t)dt = x - \int_0^x H(t)dt$$

for all valuations, $x \in [0, \omega]$.

Proof. The profit function for a bidder in the second-price sealed-bid auction is given by

$$\begin{aligned} \Pi^{\mathbf{II}}(b,x) &= G(\beta^{-1}(b))\left(\frac{\int_0^{\beta^{-1}(b)}(x-\beta(y))g(y)dy}{G(\beta^{-1}(b))}\right) \\ &\quad + \left(1 - G(\beta^{-1}(b))\right)\int_0^x (x-t)h(t)dt \\ &= \int_0^{\beta^{-1}(b)} (x-\beta(y))g(y)dy + \left(1-G(\beta^{-1}(b))\right)\int_0^x H(t)dt \end{aligned} \tag{4.3.1}$$

Assume the other $N-1$ bidders follow bidding strategy $\beta(x) = x - \int_0^x H(t)dt$. Note that $\beta(x)$ is increasing since $\beta'(x) = 1 - H(x) > 0$. Substitution yields

$$\begin{aligned} \Pi^{\mathbf{II}}(b,x) &= \int_0^{\beta^{-1}(b)} \left(x - y + \int_0^y H(t)dt\right) g(y)dy \\ &\quad + \left(1 - G(\beta^{-1}(b))\right)\int_0^x H(t)dt. \end{aligned} \tag{4.3.2}$$

It is helpful to find the inverse of the bidding function. Since

$$\beta(x) = x - \int_0^x H(t)dt$$

and

$$\beta(\beta^{-1}(x)) = x$$

the inverse is found implicitly from

$$x = \beta^{-1}(x) - \int_0^{\beta^{-1}(x)} H(t)dt. \tag{4.3.3}$$

Differentiating 4.3.2 with respect to the bid, b, gives the following first order condition.

$$\begin{aligned}
\frac{\partial \Pi(b,x)}{\partial b} &= \left(x - \beta^{-1}(b) + \int_0^{\beta^{-1}(b)} H(t)dt\right)\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(x))} \\
&\quad - \frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(x))}\int_0^x H(t)dt \\
&= \left(x - \beta^{-1}(b) + \int_0^{\beta^{-1}(b)} H(t)dt - \int_0^x H(t)dt\right)\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(x))} \\
&= 0
\end{aligned} \tag{4.3.4}$$

The first order condition reduces to

$$\begin{aligned}
0 &= x - \beta^{-1}(b) + \int_0^{\beta^{-1}(b)} H(t)dt - \int_0^x H(t)dt \\
&= x - b - \int_0^{\beta^{-1}(b)} H(t)dt + \int_0^{\beta^{-1}(b)} H(t)dt - \int_0^x H(t)dt \\
b &= x - \int_0^x H(t)dt
\end{aligned} \tag{4.3.5}$$

The second equality follows by substitution of equation 4.3.3. It remains to show that this solution is unique. The profit function is now shown to be unimodal since it is strictly concave whenever the first order condition is satisfied.

$$\begin{aligned}\frac{\partial^2 \Pi(b,x)}{\partial b^2} = &\left(x - \beta^{-1}(b) + \int_0^{\beta^{-1}(b)} H(t)dt - \int_0^x H(t)dt\right)\frac{\partial}{\partial b}\left(\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(x))}\right) \\ &- \frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(x))}\left(\frac{(1 - H(\beta^{-1}(b)))}{\beta'(\beta^{-1}(x))}\right)\end{aligned} \tag{4.3.6}$$

At the first order condition, the first term is zero and the second term is negative since $\beta'(\cdot) > 0$, $H(\cdot) < 1$, and $g(\cdot) > 0$. Since the function is concave at the first order condition, the solution in 4.3.5 is unique. □

Without the option of buying the item at a later date, each bidder would bid their valuation. When the product is available at a later date, the bidders discount their bid by the amount

$$\int_0^x (x - t)h(t)dt.$$

This term can be rewritten as $H(x)E[x - T|T < x]$, which is the probability that the fixed price is less than the valuation, x, times the expected surplus given that the fixed price is less than the valuation.

The following corollary is extremely useful for proving the equilibrium bidding strategies for first-price auctions in the upcoming sections.

Corollary 1. *For all $x \neq z, x, z \in [0, \omega]$ and distributions, $G(\cdot), H(\cdot)$, the following inequality is true.*

$$G(z)(z - x) - \int_x^z \Big(G(t) + G(z)H(t) - G(t)H(t)\Big)dt > 0. \tag{4.3.7}$$

Proof. Consider the second price auction without rebate. From the preceding theorem, if $\beta(x) = x - \int_0^x H(t)dt$, then $\Pi(\beta(x), x) - \Pi(\beta(z), x) > 0$ since $\beta(x)$ is a symmetric equilibrium strategy. It is now shown that $\Pi(\beta(x), x) - \Pi(\beta(z), x)$ is precisely the expression given in 4.3.7.

$$\begin{aligned}
\Pi(\beta(z), x) &= \int_0^{\beta^{-1}(\beta(z))} \left(x - y + \int_0^y H(t)dt \right) g(y)dy \\
&\quad + \left(1 - G(\beta^{-1}(\beta(z))) \right) \int_0^x H(t)dt \\
&= xG(z) - \int_0^z yg(y)dy + \int_0^z \int_0^y H(t)g(y)dtdy \\
&\quad + (1 - G(z)) \int_0^x H(t)dt \\
&= xG(z) - zG(z) + \int_0^z G(y)dy + \int_0^z \int_t^z H(t)g(y)dydt \\
&\quad + (1 - G(z)) \int_0^x H(t)dt \\
&= G(z)(x - z) + \int_0^z G(y)dy + \int_0^z H(t)(G(z) - G(t))dt \\
&\quad + (1 - G(z)) \int_0^x H(t)dt \\
&= G(z)(x - z) + \int_0^z G(y)dy + G(z) \int_0^z H(t)dt \\
&\quad - \int_0^z G(t)H(t)dt + (1 - G(z)) \int_0^x H(t)dt
\end{aligned}$$

Finding the desired difference,

$$\begin{aligned}
&\Pi(\beta(x), x) - \Pi(\beta(z), x) \\
&\qquad = G(z)(x - x) + \int_0^x G(y)dy + G(x) \int_0^x H(t)dt \\
&\qquad - \int_0^x G(t)H(t)dt + (1 - G(x)) \int_0^x H(t)dt \\
&\qquad - G(z)(x - z) - \int_0^z G(y)dy - G(z) \int_0^z H(t)dt \\
&\qquad + \int_0^z G(t)H(t)dt - (1 - G(z)) \int_0^x H(t)dt \\
&\qquad = G(z)(z - x) - \int_x^z \Big(G(t) + G(z)H(t) - G(t)H(t) \Big) dt.
\end{aligned} \tag{4.3.8}$$

Thus, the expression is greater than zero. □

4.4 First-Price Auction without Rebate

The item is to be auctioned with the following rules. The highest bid wins and pays the amount of the highest bid. Losing bidders have the option to buy the item at a later date for an unknown price, T. If the price paid at auction is higher than the fixed price, no rebate is given. The expected payoff for a consumer with valuation x when he bids b is given in equation 4.4.9.

$$\begin{aligned}\Pi^{\mathbf{I}}(b,x) &= G(\beta^{-1}(b))(x-b) + (1 - G(\beta^{-1}(b))\int_0^x (x-t)h(t)dt \\ &= G(\beta^{-1}(b))(x-b) + (1 - G(\beta^{-1}(b))\int_0^x H(t)dt\end{aligned} \tag{4.4.9}$$

The optimal bidding strategy is now found by solving the differential equation derived from the first order conditions. Differentiating 4.4.9 with respect to the bid, b, gives the first order condition.

$$\frac{\partial\Pi(x,b)}{\partial b} = \frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}\left(x - b - \int_0^x (x-t)h(t)dt\right) - G(\beta^{-1}(b)) = 0 \tag{4.4.10}$$

Letting $b = \beta(x)$ and rearranging 4.4.10 yields the following differential equation.

$$g(x)\left(x - \beta(x) - \int_0^x (x-t)h(t)dt\right) - \beta'(x)G(x) = 0 \tag{4.4.11}$$

Let $y = \beta(x)$.

$$g(x)\left(x - y - \int_0^x (x-t)h(t)dt\right) - G(x)y' = 0 \tag{4.4.12}$$

This is a differential equation of the form $M(x,y) + N(x,y)y' = 0$. Note that $M_y(x,y) = -g(x) = N_x(x,y)$, implying that this is an exact differential equation (Boyce and DiPrima 1992). Therefore, there exists a function, $\psi(x,y)$, such that $\psi_x(x,y) = M(x,y)$ and $\psi_y(x,y) = N(x,y)$ where $\psi(x,y) = k$ is an implicit solution to the differential equation. $\psi(x, y)$ is now found.

$$\begin{aligned}
\psi &= -yG(x) + v(x) \\
\psi_x &= -yg(x) + v'(x) \\
&= -yg(x) + g(x)\left(x - \int_0^x (x-t)h(t)dt\right) \\
v(x) &= G(x)\left(x - \int_0^x (x-t)h(t)dt\right) - \int_0^x G(t)(1-H(t))dt \\
\psi &= -yG(x) + G(x)\left(x - \int_0^x (x-t)h(t)dt\right) - \int_0^x G(t)(1-H(t))dt = k
\end{aligned}$$

To find the constant, k, note that $\beta(0) = 0 = y(0)$. Thus, $k = 0$. Solving for y gives the symmetric equilibrium bidding strategy.

$$\begin{aligned}
\beta^{\mathbf{I}}(x) &= x - \int_0^x (x-t)h(t)dt - \int_0^x \frac{G(t)(1-H(t))dt}{G(x)} \\
&= x - \int_0^x H(t)dt - \int_0^x \frac{G(t)(1-H(t))dt}{G(x)}
\end{aligned} \tag{4.4.13}$$

The result is formally proven.

Theorem 2. *In the first-price auction without rebate, a symmetric equilibrium bidding strategy is*

$$\begin{aligned}
\beta^{\mathbf{I}}(x) &= x - \int_0^x (x-t)h(t)dt - \int_0^x \frac{G(t)(1-H(t))dt}{G(x)} \\
&= x - \int_0^x H(t)dt - \int_0^x \frac{G(t)(1-H(t))dt}{G(x)}.
\end{aligned}$$

Proof. First, note that $\beta(x)$ is increasing.

$$\begin{aligned}
\beta'(x) &= 1 - H(x) - \frac{(G(x))^2(1-H(x)) - g(x)\int_0^x G(t)(1-H(t))dt}{(G(x))^2} \\
&= \frac{g(x)\int_0^x G(t)(1-H(t))dt}{(G(x))^2} > 0
\end{aligned} \tag{4.4.14}$$

Assume that the $N-1$ other bidders use the strategy as described. It is necessary to show that a bidder with valuation x has no incentive to bid as though his valuation was $z, z \neq x$. That is, $\Pi(\beta(x), x) - \Pi(\beta(z), x) > 0$. Substituting the bidding function into 4.4.9 gives

$$\begin{aligned}
\Pi(\beta(z), x) &= G(\beta^{-1}(\beta(z)))(x - \beta(z)) + (1 - G(\beta^{-1}(\beta(z))) \int_0^x H(t)dt \\
&= G(z)(x - \beta(z)) + (1 - G(z)) \int_0^x H(t)dt \\
&= G(z)\left(x - z + \int_0^z H(t)dt + \int_0^z \frac{G(t)(1 - H(t))dt}{G(z)}\right) \\
&\quad + (1 - G(z)) \int_0^x H(t)dt \\
&= G(z)(x - z) + G(z) \int_0^z H(t)dt + \int_0^z G(t)(1 - H(t))dt \\
&\quad + (1 - G(z)) \int_0^x H(t)dt.
\end{aligned} \tag{4.4.15}$$

The difference is now found.

$$\begin{aligned}
&\Pi(\beta(x), x) - \Pi(\beta(z), x) \\
&\quad = G(x)(x - x) + G(x) \int_0^x H(t)dt \\
&\quad\quad + \int_0^x G(t)(1 - H(t))dt + (1 - G(x)) \int_0^x H(t)dt \\
&\quad\quad - G(z)(x - z) - G(z) \int_0^z H(t)dt \\
&\quad\quad - \int_0^z G(t)(1 - H(t))dt - (1 - G(z)) \int_0^x H(t)dt \\
&\quad = G(z)(z - x) - G(z) \int_x^z H(t)dt - \int_x^z G(t)(1 - H(t))dt \\
&\quad = G(z)(z - x) - \int_x^z \Big(G(t) + G(z)H(t) - G(t)H(t)\Big)dt > 0
\end{aligned} \tag{4.4.16}$$

The last inequality follows from Corollary 1. Thus, the bidding strategy is a symmetric equilibrium. □

Without the option of purchasing at the fixed price, the equilibrium first-price auction bidding strategy is $\beta(x) = E[Y_1|Y_1 < x]$ (see Krishna 2002). With the option to purchase later, the symmetric equilibrium bidding strategy from above can be written as

$$\beta^{\mathbf{I}}(x) = E[Y_1|Y_1 < x] - \frac{1}{G(x)} \int_0^x \Big(G(x)H(t) - G(t)H(t)\Big)dt,$$

which is strictly less than $E[Y_1|Y_1 < x]$.

4.5 First-Price Auction with Rebate

Consider the first-price auction with rebate. The highest bid wins and pays the amount of the highest bid. Losing bidders have the option to buy the item at a later date for an unknown price. If the fixed price of the item is less than the winning bid, the winning bidder gets a rebate for the difference. The expected payoff for a consumer with valuation x when he bids b is given in equation 4.5.17.

$$\begin{aligned}\Pi^{\mathbf{I_R}}(b,x) &= G(\beta^{-1}(b))\left(x - b + \int_0^b (b-t)h(t)dt\right) \\ &\quad + (1 - G(\beta^{-1}(b))\int_0^x (x-t)h(t)dt \\ &= G(\beta^{-1}(b))\left(x - b + \int_x^b H(t)dt\right) + \int_0^x H(t)dt\end{aligned} \tag{4.5.17}$$

Differentiating 4.5.17 with respect to the bid, b, gives the first order condition.

$$\frac{\partial\Pi(x,b)}{\partial b} = \frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}\left(x - b + \int_x^b H(t)dt\right) - G(\beta^{-1}(b))(1 - H(b)) = 0 \tag{4.5.18}$$

Letting $b = \beta(x)$ and rearranging 4.5.18 yields the following differential equation.

$$g(x)\left(x - \beta(x) + \int_x^{\beta(x)} H(t)dt\right) - \beta'(x)G(x)(1 - H(\beta(x)) = 0 \tag{4.5.19}$$

Let $y = \beta(x)$.

$$g(x)\left(x - y + \int_x^y H(t)dt\right) - G(x)(1 - H(y))y' = 0 \tag{4.5.20}$$

This is a differential equation of the form $M(x, y) + N(x, y)y' = 0$. Note that $M_y(x, y) = -g(x)(1 - H(y)) = N_x(x, y)$, implying that this is an exact differential equation. Therefore, there exists a function, $\psi(x, y)$, such that $\psi_x(x, y) = M(x, y)$ and $\psi_y(x, y) = N(x, y)$ where $\psi(x, y) = k$ is an implicit solution to the differential equation. $\psi(x, y)$ is now found.

$$\begin{aligned}
\psi &= -G(x)\left(y - \int_0^y H(t)dt\right) + v(x) \\
\psi_x &= -g(x)\left(y - \int_0^y H(t)dt\right) + v'(x) \\
&= -g(x)\left(y - \int_0^y H(t)dt\right) + g(x)\left(x - \int_0^x H(t)dt\right) \\
v(x) &= G(x)\left(x - \int_0^x H(t)dt\right) - \int_0^x G(t)(1 - H(t))dt \\
\psi &= -G(x)\left(y - \int_0^y H(t)dt\right) + G(x)\left(x - \int_0^x H(t)dt\right) \\
&\quad - \int_0^x G(t)(1 - H(t))dt = k
\end{aligned}$$

To find the constant, k, note that $\beta(0) = 0 = y(0)$. Thus, $k = 0$. While it is not possible to solve for y explicitly, the optimal bidding strategy can be found implicitly.

$$\beta^{\mathbf{IR}}(x) - \int_0^{\beta^{\mathbf{IR}}(x)} H(t)dt = x - \int_0^x H(t)dt - \int_0^x \frac{G(t)(1 - H(t))dt}{G(x)} \tag{4.5.21}$$

The result is now formally proven.

Theorem 3. *In the first-price auction with rebate, a symmetric equilibrium bidding* *strategy is to bid* $\beta^{\mathbf{I_R}}(x)$ *which implicitly solves*

$$\beta^{\mathbf{I_R}}(x) - \int_0^{\beta^{\mathbf{I_R}}(x)} H(t)dt = x - \int_0^x H(t)dt - \int_0^x \frac{G(t)(1-H(t))dt}{G(x)}.$$

Proof. First, we verify that $\beta(x)$ is increasing. Taking the derivative of both sides gives the following.

$$\begin{aligned}
\beta'(x) - H(\beta(x))\beta'(x) &= 1 - H(x) \\
&\quad - \frac{(G(x))^2(1-H(x)) - g(x)\int_0^x G(t)(1-H(t))dt}{(G(x))^2} \\
\beta'(x)(1-H(\beta(x))) &= \frac{g(x)\int_0^x G(t)(1-H(t))dt}{(G(x))^2} \\
\beta'(x) &= \frac{g(x)\int_0^x G(t)(1-H(t))dt}{(G(x))^2(1-H(\beta(x)))} > 0
\end{aligned} \tag{4.5.22}$$

Assume that the $N-1$ other bidders use the strategy as described. Is is now shown that a bidder with valuation x has no incentive to bid as though his valuation was $z, z \neq x$. That is, $\Pi(\beta(x), x) - \Pi(\beta(z), x) > 0$. Substituting the bidding function into 4.5.17 gives

$$\begin{aligned}
\Pi(\beta(z), x) &= G(\beta^{-1}(\beta(z)))\left(x - \beta(z) + \int_0^{\beta(z)} H(t)dt - \int_0^x H(t)dt)\right) \\
&\quad + \int_0^x H(t)dt \\
&= G(z)\left(x - \int_0^x H(t)dt - z + \int_0^z H(t)dt\right. \\
&\quad \left. + \int_0^z \frac{G(t)(1-H(t))dt}{G(z)}\right) + \int_0^x H(t)dt \\
&= G(z)\left(x - z + \int_x^z H(t)dt\right) + \int_0^z G(t)(1-H(t))dt \\
&\quad + \int_0^x H(t)dt
\end{aligned}$$

The difference is now found.

$$
\begin{aligned}
&\Pi(\beta(x),x)-\Pi(\beta(z),x)\\
&\quad= G(x)\left(x-x+\int_x^x H(t)dt\right)+\int_0^x G(t)(1-H(t))dt+\int_0^x H(t)dt\\
&\qquad -G(z)\left(x-z+\int_x^z H(t)dt\right)-\int_0^z G(t)(1-H(t))dt-\int_0^x H(t)dt\\
&\quad= G(z)(z-x)-\int_x^z \Big(G(t)+G(z)H(t)-G(t)H(t)\Big)dt>0
\end{aligned}
$$

The last inequality follows from Corollary 1. Thus, the bidding strategy in equation 4.5.21 is a symmetric equilibrium. □

It is easily seen that bids are higher when a rebate is given. That is, $\beta^{\mathbf{I_R}}(x) > \beta^{\mathbf{I}}(x)$. It is unclear whether this bid amount is greater than the equilibrium bid in a first-price auction without the option of purchasing at a later date.

4.6 Second-Price Auction with Rebate

It is now shown that a second-price auction with rebate results in bidders truthfully revealing their valuations. A bidder will either win the object in the auction and pay the second highest bid, or if the bidder loses the auction, he will purchase the item for some unknown price at a later date if the fixed price is less than his valuation. If the payment made by the winner of the auction is greater than the fixed price, a rebate is given for the difference.

The intuition behind this format is as follows. The rebate against the fixed price addresses the issue of bid adjustment due to the existence of an outside option (future fixed price) As we explain below, the second price sealed-bid auction with rebate ensures truthful bidding in equilibrium, even if the bidders anticipate a future outside option of purchasing at a fixed price. The expected payoff to a bidder with valuation x when bidding b is given below when the $N-1$ other bidders follow bidding strategy

$\beta(x)$.

$$\begin{aligned}\Pi^{\mathbf{II_R}}(b,x) &= G(\beta^{-1}(b))\Bigg(\frac{\int_0^{\beta^{-1}(b)}(x-\beta(y))g(y)dy}{G(\beta^{-1}(b))} \\ &\quad + \frac{\int_0^{\beta^{-1}(b)}\int_0^{\beta(y)}(\beta(y)-t)h(t)g(y)dtdy}{G(\beta^{-1}(b))}\Bigg) \\ &\quad + \Big(1-G(\beta^{-1}(b))\Big)\int_0^x (x-t)h(t)dt \\ &= \int_0^{\beta^{-1}(b)}(x-\beta(y))g(y)dy \\ &\quad + \int_0^{\beta^{-1}(b)}\int_0^{\beta(y)}(\beta(y)-t)h(t)g(y)dtdy \\ &\quad + \Big(1-G(\beta^{-1}(b))\Big)\int_0^x H(t)dt\end{aligned} \tag{4.6.23}$$

Theorem 4. *The symmetric equilibrium bidding strategy for a second-price auction with rebate is* $\beta^{\mathbf{II_R}}(x) = x$.

Proof. Assume that the $N-1$ of the N bidders follow strategy $\beta(x) = x$. It is shown that this strategy is also optimal for the remaining bidder. Equation 4.6.23 is now rewritten to reflect the bidding strategies of the other $N-1$ bidders.

$$\begin{aligned}\Pi^{\mathbf{II_R}}(b,x) &= \int_0^b (x-y)g(y)dy + \int_0^b\int_0^y (y-t)h(t)g(y)dtdy \\ &\quad +\Big(1-G(b)\Big)\int_0^x H(t)dt \\ &= xG(b) - bG(b) + \int_0^b G(y)dy + \int_0^b\int_0^y H(t)g(y)dtdy \\ &\quad +\Big(1-G(b)\Big)\int_0^x H(t)dt \\ &= G(b)(x-b) + \int_0^b G(y)dy + \int_0^b\int_t^b H(t)g(y)dydt \\ &\quad +\Big(1-G(b)\Big)\int_0^x H(t)dt \\ &= G(b)(x-b) + \int_0^b G(y)dy + G(b)\int_0^b H(t)dt\end{aligned}$$

$$
\begin{aligned}
&-\int_0^b G(t)H(t)dt + \Big(1 - G(b)\Big)\int_0^x H(t)dt \\
= \;& G(b)(x-b) + \int_0^b G(y)dy + G(b)\int_x^b H(t)dt \\
&-\int_0^b G(t)H(t)dt + \int_0^x H(t)dt
\end{aligned}
$$

The first order condition is used to find an optimal bid, b, when the valuation is x.

$$
\begin{aligned}
\frac{\partial \Pi(b,x)}{\partial b} &= g(b)(x-b) - G(b) + G(b) \\
&\quad + g(b)\int_x^b H(t)dt + G(b)H(b) - G(b)H(b) \\
&= g(b)\left(x - b + \int_x^b H(t)dt\right) = 0
\end{aligned} \tag{4.6.24}
$$

Clearly, the first order condition is satisfied when $b = x$. To verify uniqueness, the profit function is shown to be strictly concave when the first order condition is satisfied. The implies that the profit function is unimodal.

$$
\begin{aligned}
\frac{\partial^2 \Pi(b,x)}{\partial b^2} &= g'(b)\left(x - b + \int_x^b H(t)dt\right) - g(b)(1 - H(b)) \\
&= -g(b)(1 - H(b)) < 0
\end{aligned} \tag{4.6.25}
$$

by the first order condition above and the fact that $H(t) \leq 1$ and $g(t) > 0$ for all $t \in [0, \omega]$. $\square$

A simple interpretation of this result is to think of the unknown fixed price as another bidder in a tradition second-price auction. The winner of the auction pays the minimum of the fixed price or the second highest bid. By offering the rebate, the firm takes the risk away from bidders, inducing them to bid their valuations. Note that equilibrium bidding strategy is not affected by the number of bidders, the distribution of bidders' valuations, or the distribution each bidder places on the unknown fixed price.

The second price sealed-bid auction with rebate is detail free in the sense that bidders can compute a bidding strategy without having to estimate the underlying consumer valuation distribution, number of bidders, distribution of the fixed price, etcetera. This is particularly important for a new product whose consumer valuation distribution is quite possibly unknown to individual consumers. The proposed rebate is, to our knowledge, novel in auction design, but from an implementation standpoint is akin to standard, post-purchase price matching guarantees common in many consumer markets, including electronics, office products, appliances, and books (Jain and Srivastava 2000, Srivastava and Lurie 2001). Note that the rebate is necessary; if it were eliminated, bidders would account for the future fixed price by bidding below their true valuation as we saw in §4.3.

While the above game-theoretical analysis can be useful for a firm wishing to solve the bid-to-willingness-to-pay inverse problem, the question of how well a given auction format actually induces truthful bidding in practice is an empirical one. While consensus has not been reached in the literature on this issue (e.g., Noussair et al. 2004), one prominent alternative, truthful auction-like mechanism used widely by experimental economists eliminates the connection between payments and other bidders' bids. Called the Becker-DeGroot-Marschak (BDM) mechanism (dating back to the seminal paper in the psychology literature by these authors, Becker et al. 1964), this mechanism accepts sealed bids and generates a random selling price at which all bids exceeding it transact. In the absence of a future fixed price option, the BDM mechanism is incentive compatible, or truthful (e.g., Kagel and Roth 1995, p79). In the proof of Theorem 4, we show that, in the presence of a fixed price option (and under the assumptions of §4.2), the BDM mechanism remains incentive compatible if a rebate is used. Thus, the rebate approach proposed in this chapter preserves incentive compatibility for both second-price (Vickrey) and BDM mechanisms, which are the two most widely studied demand-revelation mechanisms in experimental economics

(Noussair et al. 2004).

4.7 Conclusion

In this chapter, we have presented a theory on the bidding strategies for consumers who participate in auctions where there is a future option for losing bidders to purchase at an unknown fixed price. The results are intuitive in that the outside option causes bidders to reduce their bid amounts, relative to what their bids would have been in first-price and second-price auctions without the option to purchase at a fixed price.

We found that the second-price auction with rebate induced the bidders to truthfully reveal their valuations. In the second-price auction without rebate, the bidders reduced their bids by an amount that depended only upon the distribution of the fixed price. In the first-price auction with rebate and without rebate, we showed that bidders shade their bids by amounts that depend on both the distribution of the fixed price and the distribution of the valuations of the other bidders.

While Corollary 1 may not appear to instill any managerial insights, it does provide a mathematical contribution which may have wider application than that which was found in this study. The reason why this seems plausible is that the construct of Corollary 1 was found to occur naturally in the proofs of symmetric bidding strategies for three different mechanisms – the second-price auction without rebate, the first-price auction without rebate, and the first-price auction with rebate. It is possible that this construct finds itself naturally in other multi-period auction mechanisms, as well.

In this study, we have made no attempt to justify whether these mechanisms are optimal in the sense of maximizing auction revenue. The rationale for using these mechanisms is to get data for market research by reverse engineering bid data into a willingness to pay distribution. To that end, this work has provided a theoretical basis.

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Chapter 5

Conclusion

This dissertation has has used methods from decision making under uncertainty to provide important insights into the control of queueing systems and stopping time problems related to market research. We have also proven equilibrium bidding strategies for several auction formats which could conceivably be used to collect willingness to pay information from consumers.

One of our main contributions in the field of queueing theory was to combine server allocation and customer routing policy decisions into a single model. Through an extensive numerical study, we were able to prescribe which form of flexibility has the most benefit, based on the parameters of the system. We also diverge from the literature in that we study the case of server pooling where there is a loss of efficiency when the servers work together. While this phenomenon is common in actual systems, it has proven difficult to obtain analytical results, and hitherto, has remained absent from the literature.

Our work on using auctions for market research is novel in the sense that we explicitly model the cost of gaining additional information as the loss in future market potential. We provided conditions under which changes in model parameters will either encourage or discourage the continuation of gathering information prior to launching a product to the mass market. Additionally, in the second part of this dissertation, we made a contribution to the literature on stochastic orderings. In particular, we showed that a vector of dependent random variables is decreasing in convex order as more data points are obtained for a specific dependence structure and random variables with a Beta-Binomial distribution (see Proposition 7 in the Appendix of chapter 3).

Finally, we make contributions to theory of auctions in our analysis of the symmetric equilibrium bidding strategies in first and second price auctions when there will be a buying opportunity at a later date. Our focus was on obtaining the equilibrium symmetric bidding strategies, so that we can reverse engineer bids to actual willingness to pay data, rather than on revenue equivalence or optimality of the auction format for auction revenue purposes.